13) Show that using the field strength $F_{\mu\nu}$ and the 4-current J^μ we can write the Maxwell equations as

$$\partial_\nu F^{\mu\nu} = J^\mu \ , \qquad \epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0 \ .$$

solution: We need to unpack those equations by discriminating between indices being 0 or $i = 1, 2, 3$ (we use latin letters for indices running from 1 to 3). As $F^{0i} = E_i$ the first equation gives

$$\partial_i E_i = \nabla \cdot \boldsymbol{E} = J^0 = \rho \tag{0.1}$$

when $\mu = 0$. For $\mu = i$ we use $F_{i0} = -E_i$ and $F^{ij} = \epsilon_{ijk}B_k$ to find

$$\partial_0 F^{i0} + \partial_j F^{ij} = -\partial_t E_i + \epsilon_{ijk}\partial_j B_k = j^i \tag{0.2}$$

which reads

$$\nabla \times \boldsymbol{B} - \frac{\partial}{\partial t}\boldsymbol{E} = \boldsymbol{j} \tag{0.3}$$

in vector notation. These are the inhomogeneous Maxwell eqns.

Let us no unpack the homogeneous eqs. Let us first set $\mu = 0$. Then $\epsilon^{0ijk} = \epsilon_{ijk}$ and hence

$$0 = \epsilon_{ijk}\partial_i F_{jk} = \epsilon_{ijk}\partial_i \epsilon_{jkl} B_l = 2\delta_{il}\partial_i B_l = \partial_i B_i \tag{0.4}$$

i.e.

$$\nabla \cdot \boldsymbol{B} = 0. \tag{0.5}$$

Finally let $\mu = i$ in the inhomogeneous Maxwell eq. Then one of the other 3 indices must be 0 so that we can write

$$\begin{aligned} 0 &= \epsilon^{i0jk}\partial_0 F_{jk} + \epsilon^{ij0k}\partial_j F_{0k} + \epsilon^{ijk0}\partial_j F_{k0} \\ &= \epsilon^{i0jk}\partial_0 \epsilon_{jkl} B_l + 2\epsilon^{0ijk}\partial_j(-E_k) \\ &= -\epsilon_{ijk}\epsilon_{jkl}\partial_0 B_l + 2\epsilon^{0ijk}\partial_j(-E_k) \\ &= \left(-2\frac{\partial}{\partial t}\boldsymbol{B} - 2\nabla \times \boldsymbol{E}\right)_i \end{aligned} \tag{0.6}$$

14) Show that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

can be written as

$$\boldsymbol{E} = -\nabla\phi - \frac{\partial \boldsymbol{A}}{\partial t} \ , \qquad \boldsymbol{B} = \nabla \times \boldsymbol{A} \ .$$

solution: We can proceed similar as above. Let us first set $\mu = 0$ and $\nu = k$. We get

$$F_{0k} = -E_k = \partial_0 A_k - \partial_k A_0 = \frac{\partial}{\partial t} A_k + \partial_k \phi \tag{0.7}$$

where we have used $A_0 = -A^0 = -\phi$. Now let us look at the situation $\mu = i$ and $\nu = j$. We find

$$F_{ij} = \epsilon_{ijk} B_k = \partial_i A_j - \partial_j A_i \,. \tag{0.8}$$

The fastest way to understand this equation is to fix e.g. $i = 1, j = 2$. In this case we find

$$\epsilon_{12k} B_k = B_3 = \partial_1 A_2 - \partial_2 A_1 = (\nabla \times \boldsymbol{A})_3 \tag{0.9}$$

and similarly for other cases. This can also be seen by contracting the above with ϵ_{ijl} to find

$$\epsilon_{ijl}\epsilon_{ijk} B_k = \epsilon_{ijl}(\partial_i A_j - \partial_j A_i) \tag{0.10}$$

which gives

$$2B_l = 2\epsilon_{ijl}\partial_i A_j = 2(\nabla \times \boldsymbol{A})_l \tag{0.11}$$

15) Show

$$\frac{\partial}{\partial X_{a_1 \dots a_n}} (X^{b_1 \dots b_n} X_{b_1 \dots b_n}) = 2 X^{a_1 a_2 \dots a_n} \;,$$

for any tensor X with components $X_{a_1 \dots a_n}$.

solution:

We have

$$\frac{\partial}{\partial X_{a_1 \dots a_n}} (X^{b_1 \dots b_n} X_{b_1 \dots b_n}) = \frac{\partial}{\partial X_{a_1 \dots a_n}} (X_{c_1 \dots c_n} X_{b_1 \dots b_n} \eta^{c_1 b_1} \cdots \eta^{c_n b_n})$$
$$= \left(\delta^{a_1}_{c_1} \cdots \delta^{a_n}_{c_n} X_{b_1 \dots b_n} + X_{c_1 \dots c_n} \delta^{a_1}_{b_1} \cdots \delta^{a_n}_{b_n} \right) \eta^{c_1 b_1} \cdots \eta^{c_n b_n})$$
$$= X^{a_1 \dots a_n} + X^{a_1 \dots a_n} = 2 X^{a_1 \dots a_n}$$

Here are some things to ponder:

1. How do electric and magnetic fields behave under Lorentz transformations?
2. Which action reproduces the Maxwell equations?
3. What is the relationship of the potential A_μ to observable physics?