16) Consider a field theory with Lagrangian

$$\mathcal{L}_0 = -\partial_\mu\phi\overline{\partial^\mu\phi} - U(|\phi|^2) \tag{0.1}$$

and scalar potential $U(|\phi|^2) = \lambda(|\phi|^2 - a^2)^2$, with parameters $\lambda, a > 0$. The energy (or "Hamiltonian") is

$$E = \int d^3x \, \left(|\partial_0\phi|^2 + |\partial_i\phi|^2 + U(|\phi|^2)\right)$$

(a) Show that the configurations of least energy ("vacua", or "ground states") parametrize a circle in field space.

(b) Show that different vacua are related by global $U(1)$ transformations.

17) Show that

$$[D_\mu, D_\nu] = -iF_{\mu\nu} \ ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength and $D_\mu = \partial_\mu - iA_\mu$ the covariant derivative.

19) Consider "scalar electrodynamics", the field theory with Lagrangian density

$$\mathcal{L} = -\overline{D_\mu\phi}D^\mu\phi - U(|\phi|^2) - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} \ ,$$

where

$$D_\mu\phi = (\partial_\mu - iA_\mu)\phi \ , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

Show that the equations of motion (Euler-Lagrange equations) for the complex scalar field ϕ and for the real $U(1)$ gauge field A_μ are

$$D_\mu D^\mu\phi = U'(|\phi|^2)\phi \ , \qquad \partial_\nu F^{\mu\nu} = g^2 J^\mu \ ,$$

where

$$J_\mu = -i(\bar{\phi}D_\mu\phi - \phi\overline{D_\mu\phi}) \ .$$

Here are some things to ponder:

1. What is a gauge symmetry and how does it differ form an ordinary global symmetry?

2. What is a covariant derivative?