16) Consider a field theory with Lagrangian

$$\mathcal{L}_0 = -\partial_\mu\phi\overline{\partial^\mu\phi} - U(|\phi|^2) \tag{0.1}$$

and scalar potential $U(|\phi|^2) = \lambda(|\phi|^2 - a^2)^2$, with parameters $\lambda, a > 0$. The energy (or "Hamiltonian") is

$$E = \int d^3x \, \left(|\partial_0\phi|^2 + |\partial_i\phi|^2 + U(|\phi|^2)\right)$$

(a) Show that the configurations of least energy ("vacua", or "ground states") parametrize a circle in field space.

(b) Show that different vacua are related by global $U(1)$ transformations.

solution:

(a) First note that this is a sum of squares, i.e. all terms are positive definite and the integral E only vanishes if

$$\partial_0\phi = \partial_i\phi = U(|\phi|^2) = 0 \tag{0.2}$$

This means we are talking about constant fields $\phi = c$ where $U(|c|^2) = 0$. This implies that

$$|c|^2 = a^2 \quad \text{i.e.} \quad c = e^{i\theta}a\,. \tag{0.3}$$

We need to make sure these also solve the equations of motion:

$$\partial_\mu\partial^\mu\phi = U'(|\phi|^2)\phi \tag{0.4}$$

(taken from problem 19). The lhs here is clearly zero for constant fields. The rhs is

$$2\lambda(|\phi|^2 - a^2)\phi \tag{0.5}$$

which vanishes for $\phi = c = e^{i\theta}a$.

The set of vacua is hence

$$\{\phi_\theta = e^{i\theta}a|\theta = 0..2\pi\} \tag{0.6}$$

which is a circle!

(b) A global $U(1)$ acts as

$$\phi \to e^{i\beta}\phi \tag{0.7}$$

for $\beta \in 0..2\pi$. Hence we have

$$\phi_\theta \to \phi_{\theta+\beta}\,, \tag{0.8}$$

and for a given vacuum we can reach all of the others by an appropriate choice of β.

17) Show that

$$[D_\mu, D_\nu] = -iF_{\mu\nu} ,$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is the field strength and $D_\mu = \partial_\mu - iA_\mu$ the covariant derivative.

solution:

The above equation must be read as a relation between differential operators, so we need to check that it holds when applied to any function $f(x)$:

$$\begin{aligned}[D_\mu, D_\nu]f &= [(\partial_\mu - iA_\mu)(\partial_\nu - iA_\nu) - (\partial_\nu - iA_\nu)(\partial_\mu - iA_\mu)]\, f \\ &= [\partial_\mu\partial_\nu - i\partial_\mu A_\nu - iA_\mu\partial_\nu - A_\mu A_\nu - \partial_\nu\partial_\mu + i\partial_\nu A_\mu + iA_\nu\partial_\mu + A_\nu A_\mu]\end{aligned} \tag{0.9}$$

The terms with 2 or 0 derivatives cancel right away, we can use the product rule and write $\partial_\mu A_\nu f = f\partial_\mu A_\nu + A_\nu \partial_\mu f$ and likewise for the other term of this type. Then

$$\begin{aligned}[D_\mu, D_\nu]f &= -i(f\partial_\mu A_\nu + A_\nu\partial_\mu f) - iA_\mu\partial_\nu f + if(\partial_\nu A_\mu + A_\mu\partial_\nu f) + iA_\nu\partial_\mu f \\ &= -if(\partial_\mu A_\nu - \partial_\nu A_\mu) = -ifF_{\mu\nu} = -iF_{\mu\nu}f\end{aligned} \tag{0.10}$$

where it is understood that the derivatives in $F_{\mu\nu}$ do not act on f, i.e. $F_{\mu\nu}$ is just a function.

19) Consider "scalar electrodynamics", the field theory with Lagrangian density

$$\mathcal{L} = -\overline{D_\mu\phi}D^\mu\phi - U(|\phi|^2) - \frac{1}{4g^2}F_{\mu\nu}F^{\mu\nu} ,$$

where

$$D_\mu\phi = (\partial_\mu - iA_\mu)\phi , \qquad F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

Show that the equations of motion (Euler-Lagrange equations) for the complex scalar field ϕ and for the real $U(1)$ gauge field A_μ are

$$D_\mu D^\mu\phi = U'(|\phi|^2)\phi , \qquad \partial_\nu F^{\mu\nu} = g^2 J^\mu ,$$

where

$$J_\mu = -i(\bar{\phi}D_\mu\phi - \phi\overline{D_\mu\phi}) .$$

solution:

We need to use the Euler-Lagrange equation, which for a field X read

$$0 = \frac{\partial\mathcal{L}}{\partial X} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu X)} \equiv \frac{\partial\mathcal{L}}{\partial X} - \partial_0\frac{\partial\mathcal{L}}{\partial(\partial_0 X)} - \partial_i\frac{\partial\mathcal{L}}{\partial(\partial_i X)} ,$$

applied to $X = \bar{\phi}$ and $X = A_\nu$. (The Euler-Lagrange equation for $X = \phi$ is the complex conjugate of the Euler-Lagrange equation for $X = \bar{\phi}$, since the Lagrangian density is real.)

Let us first work out the equation of motion for ϕ, which is obtained by using $X = \bar{\phi}$ in the Euler-Lagrange equation above. The simplest way to proceed is perhaps to integrate by parts the kinetic term of ϕ in the action, or write the Lagrangian density as

$$\mathcal{L} = \bar{\phi} D_\mu D^\mu \phi - U(|\phi|^2) - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} + \partial_\mu(\dots) \equiv \mathcal{L}' + \partial_\mu(\dots) \ .$$

The last term is a total derivative, which integrates to a boundary (or 'surface') term in the action, which in turn does not contribute to the equations of motion (which are obtained by setting to zero the first variation of the action under any variations of the fields, see Math Phys II or the first term). Then the E-L eqn becomes $\partial\mathcal{L}'/\partial\bar{\phi} = 0$, which leads to

$$D_\mu D^\mu \phi - U'(|\phi|^2)\phi = 0 \ .$$

Alternatively, let us write down the covariant derivative explicitly:

$$\mathcal{L} = -(\partial_\mu \bar{\phi} + i A_\mu \bar{\phi})(\partial^\mu \phi - i A^\mu \phi) - U(|\phi|^2) - \frac{1}{4g^2} F_{\mu\nu} F^{\mu\nu} \ .$$

Then

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial \bar{\phi}} &= -i A_\mu (\partial^\mu \phi - i A^\mu \phi) - U'(|\phi|^2)\phi \\ \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} &= -(\partial^\mu \phi - i A^\mu \phi) \ , \end{aligned}$$

which leads to the E-L equation

$$0 = (\partial_\mu - i A_\mu)(\partial^\mu - i A^\mu)\phi - U'(|\phi|^2)\phi \equiv D_\mu D^\mu \phi - U'(|\phi|^2)\phi \ .$$

The equation of motion for the gauge field ($X = A_\nu$) is a little more involved to derive, but we can make progress if we notice that A_ν only appears inside the covariant derivatives $D_\nu \phi$ and $D_\nu \bar{\phi}$, whereas $\partial_\mu A_\nu$ only appears inside the Maxwell term, which depends on the field strength. Using the chain rule and $\partial A_\mu / \partial A_\nu = \delta^\nu_\mu$, we calculate

$$\begin{aligned} \frac{\partial \mathcal{L}}{\partial A_\nu} &= \frac{\partial \mathcal{L}}{\partial(D_\mu \phi)} \frac{\partial(D_\mu \phi)}{\partial A_\nu} + \frac{\partial \mathcal{L}}{\partial(\overline{D_\mu \phi})} \frac{\partial(\overline{D_\mu \phi})}{\partial A_\nu} = -\overline{D^\mu \phi}(-i \delta^\nu_\mu \phi) + (c.c) \\ &= i(\phi \overline{D^\nu \phi} - \bar{\phi} D^\nu \phi) \equiv J^\nu \ . \end{aligned}$$

Then we have

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} \frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} \ ,$$

where we first view $F_{\rho\sigma}$ as independent variables that the Lagrangian density depends on, and then express them as $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Let's compute the two factors separately. Being explicit with indices,

$$\begin{aligned}\frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}} &= -\frac{1}{4g^2}\eta^{\mu\alpha}\eta^{\nu\beta}\frac{\partial}{\partial F_{\rho\sigma}}(F_{\mu\nu}F_{\alpha\beta}) = -\frac{1}{4g^2}\eta^{\mu\alpha}\eta^{\nu\beta}\left(\frac{\partial F_{\mu\nu}}{\partial F_{\rho\sigma}}F_{\alpha\beta} + F_{\mu\nu}\frac{\partial F_{\alpha\beta}}{\partial F_{\rho\sigma}}\right) \\ &= -\frac{1}{4g^2}\eta^{\mu\alpha}\eta^{\nu\beta}\left(\delta^\rho_\mu\delta^\sigma_\nu F_{\alpha\beta} + F_{\mu\nu}\delta^\rho_\alpha\delta^\sigma_\beta\right) = -\frac{1}{2g^2}F^{\rho\sigma} \ .\end{aligned}$$

(Having done this exercise once, later on we will use $\frac{\partial}{\partial X_\nu}(X^\mu X_\mu) = 2X^\nu$, $\frac{\partial}{\partial X_{\rho\sigma}}(X^{\mu\nu}X_{\mu\nu}) = 2X^{\rho\sigma}$ *etc.* without further proof.) For the second factor,

$$\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = \frac{\partial}{\partial(\partial_\mu A_\nu)}(\partial_\rho A_\sigma - \partial_\sigma A_\rho) = \delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho \ .$$

Putting the previous results together, we find

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu A_\nu)} = \frac{\partial \mathcal{L}}{\partial F_{\rho\sigma}}\frac{\partial F_{\rho\sigma}}{\partial(\partial_\mu A_\nu)} = -\frac{1}{2g^2}F^{\rho\sigma}\left(\delta^\mu_\rho\delta^\nu_\sigma - \delta^\mu_\sigma\delta^\nu_\rho\right) = -\frac{1}{g^2}F^{\mu\nu} = \frac{1}{g^2}F^{\nu\mu}$$

So the equation of motion for the gauge field reads

$$0 = J^\nu - \partial_\mu\left(\frac{1}{g^2}F^{\nu\mu}\right) \quad \Longleftrightarrow \quad \partial_\mu F^{\nu\mu} = g^2 J^\nu \ .$$

Here are some things to ponder:

1. What is a gauge symmetry and how does it differ form an ordinary global symmetry?

2. What is a covariant derivative?