21) A magnetic monopole of magnetic charge m located at the origin O of three-dimensional space is described by a divergence-free magnetic field $\vec{B}$ in $\mathbb{R}^3 \backslash O$, with non-vanishing magnetic flux though the 2-sphere that surrounds the origin O:

$$\frac{1}{2\pi}\int_{S^2} \boldsymbol{B}\cdot d\boldsymbol{\sigma} = m \neq 0.$$

(a) Show that all of the above can be reformulated as the equation

$$\nabla\cdot\boldsymbol{B} = 2\pi m\ \delta^{(3)}(\vec{x})$$

in $\mathbb{R}^3$.

(b) Using that

$$\nabla\frac{1}{r} = -\frac{\boldsymbol{x}}{r^3}\ , \qquad \Delta\frac{1}{r} = -4\pi\ \delta^{(3)}(\boldsymbol{x})\ ,$$

where $r = |\boldsymbol{x}|$ and $\Delta = \nabla^2$ is the Laplacian, show that

$$\boldsymbol{B} = \frac{m}{2}\frac{\boldsymbol{x}}{r^3}$$

solves the equation in part (a).

(c) For

$$A_x^\pm = \mp\frac{m}{2}\frac{y}{r(r\pm z)}\ , \quad A_y^\pm = \pm\frac{m}{2}\frac{x}{r(r\pm z)}\ , \quad A_z^\pm = 0\ . \tag{1}$$

Show that the corresponding magnetic field is

$$\nabla\times\boldsymbol{A}^\pm = \frac{m}{2}\frac{\boldsymbol{x}}{r^3}\ . \tag{2}$$

22) Write the $A^\pm = A_x^\pm dx + A_y^\pm dy + A_z^\pm dz$ in (1) in spherical coordinates as

$$\boldsymbol{A} = \frac{m}{2}(\pm 1 - \cos\theta)d\varphi$$

using $dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi$ and similarly for dy.

23) The energy stored in electromagnetic fields is

$$\tfrac{1}{2}\int d^3x(\boldsymbol{E}^2 + \boldsymbol{B}^2)\,.$$

Show that the energy of the magnetic monopole solution (2) is infinite. How about an electric monopole?

Here are some things to ponder:

1. How can we get away with defining a magnetic monopole in a $U(1)$ gauge theory?
2. How would you go about defining a non-abelian gauge theory?