24) By considering infinitesimal gauge transformations ($|\alpha^a| \ll 1$)

$$g = e^{i\alpha^a t_a} \equiv e^{i\alpha} = 1 + i\alpha + O(\alpha^2) \tag{1.1}$$

and Taylor expanding finite gauge transformations to leading order in $\alpha \in \mathfrak{g} = \mathrm{Lie}(G)$, show that the **infinitesimal gauge variations** of the fields are

$$\begin{aligned} \delta_\alpha \phi &= i\alpha\phi \\ \delta_\alpha A_\mu &= i[\alpha, A_\mu] + \partial_\mu \alpha \\ \delta_\alpha F_{\mu\nu} &= i[\alpha, F_{\mu\nu}] \ , \end{aligned} \tag{1.2}$$

where $\phi \mapsto \phi + \delta_\alpha \phi$ and so on to leading order.

26) Consider a field ϕ in the adjoint representation, with components ϕ^a, where $a = 1, \ldots, \dim\ \mathfrak{g}$.

(a) Show that

$$(A_\mu \phi)^a = i f_{bc}{}^a A^b_\mu \phi^c \tag{1.3}$$

and similarly for $(F_{\mu\nu}\phi)^a$.
[Hint: we worked out the matrices defining the adjoint representation in problem 29 of Michaelmas term, but wrote group elements as $e^{\alpha^a \hat{t}_a}$ instead of the physics convention $e^{i\alpha^a t_a}$ used here]

(b) Let $\Phi := \phi^a t_a$, and $A_\mu = A^a_\mu t_a$, $F_{\mu\nu} = F^a_{\mu\nu} t_a$ as usual. Show that

$$(A_\mu \phi)^a t_a = [A_\mu, \Phi] \tag{1.4}$$

and similarly for $F_{\mu\nu}\phi$. Show that therefore

$$\begin{aligned} D_\mu \Phi &= \partial_\mu \Phi - i[A_\mu, \Phi] \\ [D_\mu, D_\nu]\Phi &= -i[F_{\mu\nu}, \Phi] \ . \end{aligned} \tag{1.5}$$

Here are some things to ponder:

1. How are convariant derivative and field strength defined for a non-abelian gauge theory?
2. What is the impact of charged matter in different representations?