24) By considering infinitesimal gauge transformations ($|\alpha^a| \ll 1$)

$$g = e^{i\alpha^a t_a} \equiv e^{i\alpha} = 1 + i\alpha + O(\alpha^2) \tag{1.1}$$

and Taylor expanding finite gauge transformations to leading order in $\alpha \in \mathfrak{g} = \mathrm{Lie}(G)$, show that the **infinitesimal gauge variations** of the fields are

$$\begin{aligned} \delta_\alpha \phi &= i\alpha\phi \\ \delta_\alpha A_\mu &= i[\alpha, A_\mu] + \partial_\mu \alpha \\ \delta_\alpha F_{\mu\nu} &= i[\alpha, F_{\mu\nu}] \ , \end{aligned} \tag{1.2}$$

where $\phi \mapsto \phi + \delta_\alpha \phi$ and so on to leading order.

solution: By construction, these have to give us the associated Lie algebra representations (but now space-time dependent): Expanding to linear order

$$\delta\phi = e^{i\alpha}\phi - \phi \simeq (1 + i\alpha)\phi - \phi = i\alpha\phi \tag{1.3}$$

$$\begin{aligned} \delta A_\mu &= e^{i\alpha}(A_\mu + i\partial_\mu)e^{-i\alpha} - A_\mu \\ &\simeq (1 + i\alpha)(A_\mu + i\partial_\mu)(1 - i\alpha) - A_\mu \\ &\simeq i\alpha A_\mu - iA_\mu\alpha + \partial_\mu\alpha = i[\alpha, A_\mu] + \partial_\mu\alpha \, . \end{aligned} \tag{1.4}$$

where we have used that $\alpha\partial_\mu\alpha \ll \partial_\mu\alpha$. Finally

$$\begin{aligned} \delta F_{\mu\nu} &= e^{i\alpha}F_{\mu\nu}e^{-i\alpha} - F_{\mu\nu} \\ &\simeq (1 + i\alpha)F_{\mu\nu}(1 - i\alpha) - F_{\mu\nu} \\ &\simeq i\alpha F_{\mu\nu} - iF_{\mu\nu}\alpha = i[\alpha, F_{\mu\nu}] \, . \end{aligned} \tag{1.5}$$

26) Consider a field ϕ in the adjoint representation, with components ϕ^a, where $a = 1, \ldots, \dim\ \mathfrak{g}$.

(a) Show that

$$(A_\mu\phi)^a = i f_{bc}{}^a A_\mu^b \phi^c \tag{1.6}$$

and similarly for $(F_{\mu\nu}\phi)^a$.
[Hint: we worked out the matrices defining the adjoint representation in problem 29 of Michaelmas term, but wrote group elements as $e^{\alpha^a \hat{t}_a}$ instead of the physics convention $e^{i\alpha^a t_a}$ used here]

(b) Let $\Phi := \phi^a t_a$, and $A_\mu = A_\mu^a t_a$, $F_{\mu\nu} = F_{\mu\nu}^a t_a$ as usual. Show that

$$(A_\mu\phi)^a t_a = [A_\mu, \Phi] \tag{1.7}$$

and similarly for $F_{\mu\nu}\phi$. Show that therefore

$$\begin{aligned} D_\mu\Phi &= \partial_\mu\Phi - i[A_\mu, \Phi] \\ [D_\mu, D_\nu]\Phi &= -i[F_{\mu\nu}, \Phi] \ . \end{aligned} \tag{1.8}$$

solution:

(a) In problem 29 MM we have seen that the matrices defining the adjoint representation are

$$\rho(\hat{t}_a)^b{}_c \equiv (\hat{t}_a^{adj})^b{}_c = f_{ac}{}^b \tag{1.9}$$

but then we did not include the i in the exponent and simply defined $g = e^{\hat{t}_a \alpha^a}$ as well as $[\hat{t}_a, \hat{t}_c] = f_{ac}{}^b \hat{t}_b$, whereas using the physics conventions with $g = e^{it_a\alpha^a}$. Clearly $\hat{t}_a = it_a$ which implies $[t_a, t_c] = if_{ac}{}^b t_b$. Using $\hat{t}_a = it_a$ we find

$$(t_a^{adj})^b{}_c = if_{ac}{}^b \tag{1.10}$$

and hence

$$(A_\mu\phi)^a = A_\mu^b (t_b^{adj})^a{}_c \phi_c = A_\mu^b (t_b)^a{}_c \phi_c = if_{bc}{}^a A_\mu^b \phi^c \,. \tag{1.11}$$

(b) Using part (a) we find

$$(A_\mu\phi)^a t_a = if_{bc}{}^a A_\mu^b \phi^c t_a = A_\mu^b \phi^c [t_b, t_c] = [A_\mu, \phi] \tag{1.12}$$

and repeating part (a) for $F_{\mu\nu}$ gives

$$(A_\mu\phi)^a = if_{bc}{}^a F_{\mu\nu}^b \phi^c \,. \tag{1.13}$$

so that the same computation shows that

$$(F_{\mu\nu}\phi)^a t_a = if_{bc}{}^a F_{\mu\nu}^b \phi^c t_a = F_{\mu\nu}^b \phi^c [t_b, t_c] = [F_{\mu\nu}, \phi] \tag{1.14}$$

Now we can work out

$$D_\mu\Phi = D_\mu\phi^a t_a = \partial_\mu\Phi - i(A_\mu\phi)^a t_a = \partial_\mu\Phi - i[A_\mu, \Phi] \tag{1.15}$$

as well as

$$[D_\mu, D_\nu]\Phi = [D_\mu, D_\nu]\phi^a t_a = (F_{\mu\nu}\phi)^a t_a = [F_{\mu\nu}, \Phi] \tag{1.16}$$

Here are some things to ponder:

1. How are convariant derivative and field strength defined for a non-abelian gauge theory?
2. What is the impact of charged matter in different representations?