Problem class 1

1) (a) Show that $SO(3)$ is a group using matrix multipication as the group composition.

(b) Verify that acting with $g \in SO(3)$ on a vector $\boldsymbol{v} \in \mathbb{R}^3$ as $\boldsymbol{v} \to g\boldsymbol{v}$ implies that the lengh of $\boldsymbol{v}$ stays invariant.

(c) For a matrix $g = e^\gamma$, what conditions do we need to put on γ such that $g \in SO(3)$?

(d) The group $O(3)$ is the group of matrices g which map a vector $\boldsymbol{v} \in \mathbb{R}^3$ to

$$\boldsymbol{v} \mapsto S\boldsymbol{v} \tag{0.0.1}$$

such that the inner form on $\mathbb{R}^3$, $\boldsymbol{v} \cdot \boldsymbol{v} = \sum_i v_i^2$, stays invariant.

For a matrix g in the group $O(3)$, show that $\det g = \pm 1$.

Solution:

(a) The definition says $g^{-1} = g^T$ and $\det g = 1$.

Assume $g \in SO(3)$. Then also $g^T = g^{-1} \in SO(3)$: if $g^T g = \mathbb{1}$ then also $(g^T)^T g^T = \mathbb{1}$. Furthermore $\det g^T = \det g = 1$.

Clearly $\mathbb{1} \in SO(3)$ and group multiplication is associative.

Finally, if $g, g' \in SO(3)$ we have

$$(gg')^{-1} = g'^{-1} g^{-1} = g'^T g^T = (gg')^T \tag{0.0.2}$$

and

$$\det gg' = \det g \det g' = 1\,. \tag{0.0.3}$$

(b) Let $\boldsymbol{v}' = g\boldsymbol{v}$. Then

$$\begin{aligned}\text{length}^2(\boldsymbol{v}') &= \boldsymbol{v}' \cdot \boldsymbol{v}' = v_i' v_i' = g_{ij} v_j g_{ik} v_k = v_j g^T_{ji} g_{ik} v_k = \boldsymbol{v}^T g^T g \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{v} \\ &= \text{length}^2(\boldsymbol{v})\end{aligned} \tag{0.0.4}$$

(c) We have $g^T = (e^\gamma)^T = e^{\gamma^T} = g^{-1} = e^{-\gamma}$. Hence $\gamma^T = -\gamma$. Repeating the same steps as done in the proof given in the lecture for $SU(2)$ shows that

$$1 = \det g = \det e^\gamma = e^{\text{tr}\gamma} \tag{0.0.5}$$

so that we need the trace of γ to vanish.

(d) Recall $O(3)$ is the group of 3×3 matrices with $g^{-1} = g^T$. Using $g^T g = \mathbb{1}$ we have that $1 = \det \mathbb{1} = \det g^T g = \det g^T \det g = (\det g)^2$.

This implies that $SO(3)$ has two disjoint components, one with $\det g = 1$ and one with $\det g = -1$. As the determinant is a continuous function of the components of the matrix, there is no way we there is a continuous path that takes us from matrices with $\det = -1$ to those of $\det = +1$. Hence $O(3)$ has two connected components. One of these (the one with the $+$) containes the identity and is a subgroup called $SO(3)$. The other one (the one with the $-$) does not contain the identity and is hence not a subgroup.

2) Let

$$F(g) : M_{\boldsymbol{v}} \to g M_{\boldsymbol{v}} g^\dagger =: F(g)[M_{\boldsymbol{v}}] \tag{0.0.6}$$

where $g \in SU(2)$ and

$$M_{\boldsymbol{v}} = \begin{pmatrix} v_3 & v_1 - i v_2 \\ v_1 + i v_2 & -v_3 \end{pmatrix} . \tag{0.0.7}$$

with $\vec{v} = (v_1, v_2, v_3) \in \mathbb{R}^3$. Show that $F(g)$ is a group homomorphism from $SU(2)$ to $SO(3)$.

Solution:

First of all, you might wonder if the map $F(g)$ preserves the structure of the matrix $M_{\boldsymbol{v}}$. Note that $M_{\boldsymbol{v}}$ is the most general complex 2×2 matrix with the property that $M_{\boldsymbol{v}}^\dagger = M_{\boldsymbol{v}}$ and $tr\ M_{\boldsymbol{v}} = 0$. Under $F(g)$ it is mapped to $g M_{\boldsymbol{v}} g^\dagger$ which also obeys

$$\left(g M_{\boldsymbol{v}} g^\dagger\right)^\dagger = (g^\dagger)^\dagger M_{\boldsymbol{v}}^\dagger g^\dagger = g M_{\boldsymbol{v}} g^\dagger \tag{0.0.8}$$

and

$$tr\ \left(g M_{\boldsymbol{v}} g^\dagger\right) = g_{ij} (M_{\boldsymbol{v}})_{jk} g^\dagger_{ki} = g^\dagger_{ki} g_{ij} (M_{\boldsymbol{v}})_{jk} = tr\ \left(g^\dagger g M_{\boldsymbol{v}}\right) = tr M_{\boldsymbol{v}} = 0 \tag{0.0.9}$$

as $g^\dagger g = \mathbb{1}$ as $g \in SU(2)$. Nexe notice that $F(g)$ is a linear map on the coords of v_i of $\mathbb{R}^3$ as

$$g(M_{\boldsymbol{v}} + M_{\boldsymbol{v}'}) g^\dagger = g M_{\boldsymbol{v}} g^\dagger + g M_{\boldsymbol{v}'} g^\dagger \tag{0.0.10}$$

This means that we could write F as

$$F(g) : \boldsymbol{v} \to \Phi(g) \boldsymbol{v} \tag{0.0.11}$$

for some 3×3 matrix $\Phi(g)$. We are not going to work out $\Phi(g)$ explicitely for a general g here, but note that we have worked this out in one instance in the lectures, where we showed that

$$g = e^{i\theta\sigma_3} \to \Phi(g) = \begin{pmatrix} \cos 2\theta & \sin 2\theta & 0 \\ -\sin 2\theta & \cos 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.0.12}$$

Next we observe that $\boldsymbol{v} \cdot \boldsymbol{v} = -\det M_{\boldsymbol{v}}$, which is mapped under F as follows:

$$\det M_{\boldsymbol{v}} \to \det g M_{\boldsymbol{v}} g^{\dagger} = \det g \det M_{\boldsymbol{v}} \det g^{-1} = \det M_{\boldsymbol{v}} \,. \tag{0.0.13}$$

Hence this map leaves $\boldsymbol{v}^2$ invariant. As it is also a linear map, it must hence map to $O(3)$. The map $F(g)$ is a continuous map which maps $\mathbb{1} \in SU(2)$ to $\mathbb{1} \in O(3)$. As $SU(2)$ is connected (it is S^3), it only maps to the elements of $O(3)$ connected with the identity, i.e. to $SO(3)$.

For each $g \in SU(2)$ we hence have a way to assign an element of $SO(3)$ by sending g to $F(g)$, or equivalently $\Phi(g)$. And even better, composing two maps in $SO(3)$ is the same as composing two elements in $SU(2)$:

$$F(g'g)(M_{\boldsymbol{v}}) = g'gM_{\boldsymbol{v}}g^{\dagger}g'^{\dagger} = F(g')\left[F(g)[M_{\boldsymbol{v}}]\right] \tag{0.0.14}$$

so that F is a group homomorphism.