Problem Class 2

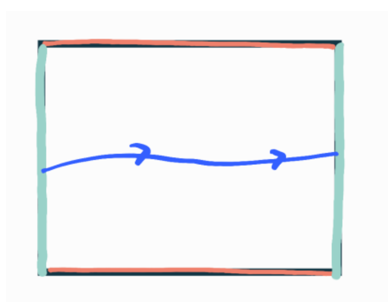
Problem 1: Patches on the torus. A two-torus T^2 is defined to be $\mathbb{R}^2$ where we identify x $x+1$ and y $y+1$ for all x and y.

a) Show that T^2 is a differentiable manifold and describe coordinate patches.

b) What is the minimal number of coordinate patches you can find?

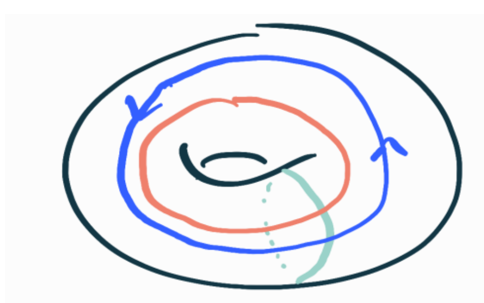
solution:

a) First of all, let's try to understand the definition of T^2 just given. Wherever we are in $\mathbb{R}^2$, we can use the identifications to make sure (x, y) are both in the interval $[0, 1]$, so that we just have a box instead of $\mathbb{R}^2$. The torus is not just a box, as e.g. the point $(1, y)$ is identified with $(0, y)$, as are $(x, 1)$ and $(x, 0)$. Hence the torus is a box with opposite edges identified and we can draw it like this:



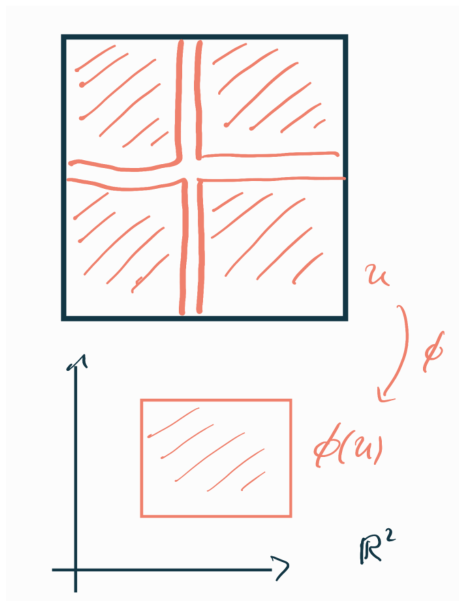
Note that the blue line is in fact a closed loop.

We can also draw the torus as a donut as follows

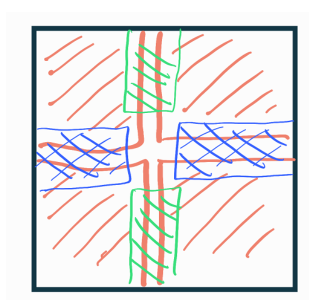


where I have shown what the red/green lines so that you can see how the square is folded into a donut. Also shown is the closed path drawn in blue above.

Now we want to describe coordinate patches from T^2 to $\mathbb{R}^2$. Let me start by drawing one patch U which includes everything except a cross in the middle. Note that U is a rectangle due to the identifications.

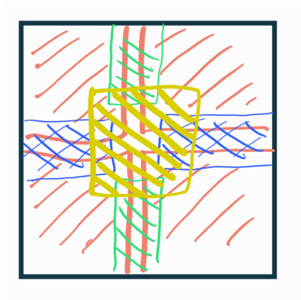


Now we can continute to cover those parts of T^2 not in U, e.g. by the green and blue stripes shown below.



For each one of these patches, we can map them to a copy of $\mathbb{R}^2$ as a homeomorphism.

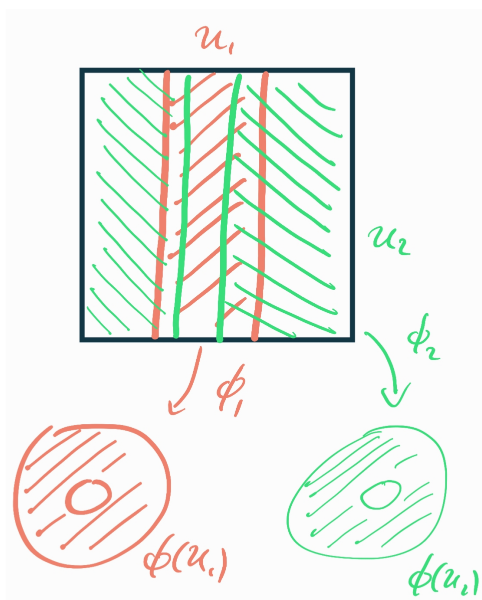
Finally, there is only a little bit in the middle not covered, and we can achieve this by mapping the yellow region to another copy of $\mathbb{R}^2$.



We are now finished and have covered T^2 by four coordinate charts. For each there is a homeomorphism to R^2 and the coordinate changes are smooth maps. We have done this in a pictorial fashion, but if you like, try to work this out explicitely.

b) As shown above, we have managed to cover T^2 by four coordinate charts, and one may wonder if we can get away with less. The image below shows how this can be accomplished using just two charts U_1 and U_2. Each of these looks like a cylinder which can be mapped by a homeomorphism to an open disc in $\mathbb{R}^2$ with a hole in the middle. The overlap $U_1 \cap U_2$ is two copies of a cylinder.

Equally well, you can think of these patches as covering a donut from above and below.



Problem 2: Describe the tangent space of $SO(3)$ at the identity.

solution:

We can approach this in two ways. First, let me construct some paths in $SO(3)$ and then use these to find an expression for tangent vectors. Here is an element of $SO(3)$:

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.1}$$

As this is in $SO(3)$ for any ϕ, we can make this into a path which I will call Γ_3 (as this is a rotation around the x_3 axis) by simply relabelling ϕ into t which is in some interval $t \in (-1, 1)$:

$$\Gamma_3 : t \mapsto \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.2}$$

It is important that this contains $t = 0$, as this is where we reach the identity, $\Gamma_3(0) = \mathbb{1}$. We can now work out the associated tangent vector as

$$T_{\mathbb{1}}(\Gamma_3) = \frac{\partial}{\partial t} \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.3}$$

Note that we could have also used the description of a patch of $SO(3)$ using coordinates to describe the tangent vector, but I chose here to use the form above instead. In the end, tangent vectors are geometrical objects, and we are free to explicitely describe them in different ways (see also problem 3 below).

We can now repeat the same for rotations around the x_1 or x_2 axis. Denoting the associated paths by Γ_1 and Γ_2 we find

$$T_{\mathbb{1}}(\Gamma_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad T_{\mathbb{1}}(\Gamma_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{0.4}$$

By a result from the lectures these are supposed to form a vector space of dimension 3. Indeed the three tangent vectors we have found span a vector space of dimension 3 using addition of matrices (the vector space of antisymmetric real 3×3 matrices), so that we can conclude that

$$T_{\mathbb{1}}SO(3) = \left\{\gamma | \gamma^T = -\gamma\right\} \tag{0.5}$$

We can recover the same result as follows: we have already seen that writing

$$g = e^{\gamma} \tag{0.6}$$

for $g \in SO(3)$ implies that $\gamma^T = -\gamma$, and any such anti-symmetric γ gives us something in $SO(3)$ upon exponentiating. Hence we can write a path

$$\Gamma_\gamma : t \mapsto e^{t\gamma} \, , \qquad t \in (-1,1)\, . \tag{0.7}$$

for any such γ. Note that multiplying an anti-symmetric matrix by a real number gives another anti-symmetric matrix, so $e^{t\gamma} \in SO(3)$ for all real t. Each such path passes through the identity for $t = 0$, so we find

$$T_{\mathbb{1}}(\Gamma_\gamma) = \frac{\partial}{\partial t}\, e^{t\gamma}\Big|_{t=0} = \gamma\, . \tag{0.8}$$

Hence the set of all tangent vectors is found again to be

$$T_{\mathbb{1}} SO(3) = \left\{\gamma | \gamma^T = -\gamma\right\}\, . \tag{0.9}$$

Problem 3:

a) Find the tangent space of a 2-sphere S^2 at the North pole using the coordinates of stereographic projection.

b) Find the tangent space of a 2-sphere S^2 at the North pole using the embedding of S^2 in $\mathbb{R}^3$.

solution:

Our sphere is $x^2 + y^2 + z^2 = 1$. Let us pick the north pole as the point $(0,0,1)$ (any other point has equal rights to be called the north pole but I will choose that one). We can pick a path

$$S_\phi : t \mapsto \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} = \begin{pmatrix} \sin(\phi)\sin(t) \\ \cos(\phi)\sin(t) \\ \cos(t) \end{pmatrix} \tag{0.10}$$

for t in some interval containing $t = 0$ (where we pass through $(0,0,1)$) and for any ϕ.

a) Let us work out the tangent vector using the coordinates

$$\boldsymbol{\varphi}_+ = \begin{pmatrix} \varphi_{+1} \\ \varphi_{+2} \end{pmatrix} = \frac{1}{1+x} \begin{pmatrix} y \\ z \end{pmatrix} \tag{0.11}$$

Our path is written in this coordinates as

$$S_\phi : t \mapsto \begin{pmatrix} \varphi_{+1}(t) \\ \varphi_{+2}(t) \end{pmatrix} = \frac{1}{1+\sin\phi\sin t} \begin{pmatrix} \cos\phi\sin t \\ \cos t \end{pmatrix} \tag{0.12}$$

and the associated tangent vector is

$$\begin{aligned} \frac{\partial}{\partial t} \begin{pmatrix} \varphi_{+1}(t) \\ \varphi_{+2}(t) \end{pmatrix}\Bigg|_{t=0} &= \frac{-\cos t\sin\phi}{1+\sin\phi\sin t} \begin{pmatrix} \cos\phi\sin t \\ \cos t \end{pmatrix} + \frac{1}{1+\sin\phi\sin t} \begin{pmatrix} \cos\phi\cos t \\ -\sin t \end{pmatrix}\Bigg|_{t=0} \\ &= \begin{pmatrix} \cos\phi \\ \sin\phi \end{pmatrix} \end{aligned} \tag{0.13}$$

By considering the paths $S_\phi(rt)$ for a real r we get

$$\begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix} \tag{0.14}$$

which gives all of $\mathbb{R}^2$ when considering all possible r and ϕ. This is the tangent vector expressed in local coordinates $\boldsymbol{\varphi}_+$. If we choose a different coordinate system, such as $\boldsymbol{\varphi}_-$, we will find a different expression.

b) Using the paths above

$$S_\phi : \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} \sin(\phi)\sin(t) \\ \cos(\phi)\sin(t) \\ \cos(t) \end{pmatrix} \right\} \tag{0.15}$$

we can just stay in $\mathbb{R}^3$ with coords x, y, z and work out the tangent vector in this description. We get

$$\frac{\partial}{\partial t} S_\phi(t)\Bigg|_{t=0} = \frac{\partial}{\partial t} \begin{pmatrix} \sin(\phi)\sin(t) \\ \cos(\phi)\sin(t) \\ \cos(t) \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} \sin(\phi) \\ \cos(\phi) \\ 0 \end{pmatrix} . \tag{0.16}$$

As we can make any choice for ϕ and furthermore consider $S_\phi(rt)$ for any real r, we get anything of the form

$$\begin{pmatrix} r\sin(\phi) \\ r\cos(\phi) \\ 0 \end{pmatrix} . \tag{0.17}$$

so these span a whole of $\mathbb{R}^2$ which is now given as sitting in $\mathbb{R}^3$.