Problem Class 3

Problem 1: Show that the Lie algebra of $SO(3)$ has the same structure constants as the Lie algebra of $SU(2)$ in an appropriate basis.

solution:

We already worked out the tangent space of $SO(3)$ at the identity in problem class 2.

For the path

$$s_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} \tag{0.1}$$

in $SO(3)$ we cross $\mathbb{1}$ for $t = 0$. We can compute the associated tangent vector

$$\ell_1 \equiv T_{\mathbb{1}}(s) = \frac{\partial}{\partial t} s(t)|_{t=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{0.2}$$

After permuting the different directions we also get the Lie algebra elements from the corresponding paths.

$$\ell_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \ell_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.3}$$

Recall problem class 2, where we showed that for $g \in SO(3)$ we can write

$$g = e^{\gamma} \tag{0.4}$$

with γ a real matrix s.t. $\gamma^T = -\gamma$. We can hence find the same result by considering the path

$$s_\gamma(t) = e^{t\gamma} \tag{0.5}$$

for any γ with $\gamma^T = -\gamma$. We compute

$$T_{\mathbb{1}}(s_\gamma) = \frac{\partial}{\partial t} s_\gamma(t)|_{t=0} = \gamma. \tag{0.6}$$

The three matrices above are the most general matrices which obey $\gamma^T = -\gamma$.

A direct computation shows that they satisfy

$$[\ell_i, \ell_j] = \epsilon_{ijk} \ell_k \, . \tag{0.7}$$

Recall that the Lie algebra of $SU(2)$ has basis vectors $i\sigma_k$ for $k = 1, 2, 3$ and the σ_k obey

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \,. \tag{0.8}$$

which is the same except for the factor of $2i$. Letting $\sigma_k = 2i\hat{\sigma}_k$ we get

$$[\hat{\sigma}_i, \hat{\sigma}_j] = \epsilon_{ijk}\hat{\sigma}_k \,. \tag{0.9}$$

which is the same algebra as the one of $SO(3)$. Note that it does not matter which matrices we write ! The whole structure is that of a vector space and map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the commutator $[.,.]$, which can be summarized by structure constants for any given basis.

What we have shown here is that both Lie algebras real 3-dimensional, and we can choose a basis where the structure constants are the same.

On the one hand, this result might not be unexpected as $SU(2)$ and $SO(3)$ are 'the same' in the vicinity of the identity: recall there is a 2-1 map from $SU(2)$ to $SO(3)$ which send $g, -g$ to the same element in $SO(3)$. To work out the Lie algebra we restrict ourselves to a small open set containing $\mathbb{1}_{SU(2)}$, which is then mapped bijectively to a small open set containing $\mathbb{1}_{SO(3)}$.

On the other hand, you might find it unsettling that 'the same' Lie algebra can give different groups. More concretely, one way to see what is going on is to observe that we get $SU(2)$ when exponentiating (i times the) Pauli matrices, whereas we get elements $SO(3)$ when exponentiating the matrices found above. It hence matters how we 'represent' this abstract thing that is a Lie algebra. (we will properly define representations of Lie algebras later).

Problem 2: Decide if the following define representations

1. Let $g \in U(2)$ act on $z \in \mathbb{C}$ as

$$z \mapsto (\det g)z \tag{0.10}$$

2. Let $g \in SO(3)$ act on $\mathbb{R}^3$ by letting

$$\vec{v} \to r(g)\vec{v} = g^2\vec{v}$$

   for $\vec{v} \in \mathbb{R}^3$.

solution:

1. In the first case we have shown that this is a group homomorphism from $U(2)$ to $U(1)$ in problem 8. The same proof shows that this is in fact a

group homomorphism from $U(2)$ to $GL(1,\mathbb{C}) = \mathbb{C}^*$, which is not surprising as we are simply mapping to the $U(1)$ subgroup of $GL(1,\mathbb{C}) = \mathbb{C}^*$.

Let us be more explicit and denote the map from $U(2)$ to $\mathbb{C}^*$ by f, i.e. $f(g) = \det g$. Both groups are multiplicative, so the only thing we need to check is that $f(gh) = f(g)f(h)$ which follows as

$$f(gh) = \det gh = \det g \det h = f(g)f(h)\,. \tag{0.11}$$

2. For this to be a representation, we need the map $f : g \mapsto g^2$ to be a group homomorphism from $SO(3)$ to $GL(3,\mathbb{R})$. Lets work out

$$f(gh) = (gh)^2 = ghgh \tag{0.12}$$

and

$$f(g)f(h) = g^2h^2\,. \tag{0.13}$$

As $[g,h] \neq 0$ for $g,h \in SO(3)$, these two expressions are not the same, so this is not a representation.

Problem 3: The adjoint action representation defines a linear map $r(g)$ acting on $\mathfrak{g}$ and as such can be written as a matrix M acting on a column vector after choosing a basis for $\mathfrak{g}$. Make this explicit for

$$g = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \in SU(2)\,. \tag{0.14}$$

Is the adjoint representation faithful?
solution: We have to work out the adjoint action on $\mathfrak{su}(2)$ in detail. We can write

$$\gamma = \sum_j i\alpha_j\sigma_j = i\begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_3 \end{pmatrix} \tag{0.15}$$

for three real numbers α_j. Note that using $i\sigma_j$ as a basis of $\mathfrak{su}(2)$, we could also represent γ as a column vector $(\alpha_1, \alpha_2, \alpha_3)$.

This is mapped to

$$i\begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_3 \end{pmatrix} \to i\begin{pmatrix} \alpha_3 & e^{2i\phi}(\alpha_1 - i\alpha_2) \\ e^{-2i\phi}(\alpha_1 + i\alpha_2) & \alpha_3 \end{pmatrix}\,. \tag{0.16}$$

The action on the α_j is hence

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \to \begin{pmatrix} \cos 2\phi & \sin 2\phi & 0 \\ -\sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \equiv M \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{0.17}$$

Note that we did exactly the same computation already in example 1.4. ! There we also realized that both g and $-g$ are mapped to the same M, so the adjoint of $SU(2)$ is not injective. Note that the same applies to the adjoint of any group (if $-g \in G$ for $g \in G$):

$$(-g)\gamma(-g)^{-1} = (-1)^2 g\gamma g^{-1} = g\gamma g^{-1} \tag{0.18}$$