Problem Class 4

Problem 1: The group $SL(2,\mathbb{C})$ is the group of complex invertible 2×2 matrices g with $\det g = 1$

1. Find the Lie algebra of $\mathfrak{sl}(2,C)$.
2. Show that the complex irreducible representations of $\mathfrak{sl}(2,C)$ are equivalent to the irreducible representations of $\mathfrak{su}(2)$.
3. Show that any irreducible representation of $SU(2)$ also gives rise to a irreducible representation of $\mathfrak{su}(2)$.

solution:

1. Note that writing
$$g = e^{i\alpha_j\sigma_j} \tag{0.1}$$
gives us an element in $\mathfrak{sl}(2,\mathbb{C})$ for any $\alpha \in \mathbb{C}^3$. These are all invertible and have $\det g = 1$, and are the most general such matrices you can write. We can then write down a path
$$g = e^{it\alpha_j\sigma_j} \quad \text{for} \ \ t = -a...a \tag{0.2}$$
which tells us that
$$\mathfrak{sl}(2,\mathbb{C}) \supset \{i\alpha_j\sigma_j | \alpha \in \mathbb{C}^3\}\,. \tag{0.3}$$
Note that the dimension of $SL(2,\mathbb{C})$ as a manifold is 6: a complex 2×2 matrix has 4×2 real entries and $\det g = 1$ are two real conditions on those (note that $\det g \in \mathbb{C}$). As the set above is also a real vector space of dimension 6, it must be the whole of the Lie algebra.

   We can express the commutators of elements of this vector space in terms of the basis vectors, which are the Pauli matrices σ_j:
$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{0.4}$$
so we are done.

2. As a representation maps elements of the Lie algebra linearly to matrices $\in \mathfrak{gl}(V)$, we can write
$$\rho(\gamma) = \rho(it\boldsymbol{\alpha\sigma}) = \sum_j i\alpha_j\rho(\sigma_j) \tag{0.5}$$

i.e. we only need to know how the Pauli matrices are represented for both cases. This means we can go back and forth between representations of these two groups by changing if we let the α_i be real or complex.

As we have complex representations, irreducibility means there should not be non-trivial invariant **complex** subspaces.

If there is an invariant subspace W for any of the two cases, then $\rho(\sigma_i)\boldsymbol{w} \in W$ for all $w \in W$ as a special case. Conversely, if $\rho(\sigma_i)\boldsymbol{w} \in W$ for all $w \in W$, then $\sum_j i\alpha_j \rho(\sigma_j)\boldsymbol{w} \in W$ for all $w \in W$. Hence the condition for (ir-)reducibility is the same.

3. We can prove this by a contradiction. Assume that r is irreducible and assume that ρ is reducible. As ρ is reducible, we can choose a basis of the vector space V these both act on where $\rho(\gamma)$ is block-diagonal for all $\gamma \in \mathfrak{su}(2)$. But then $e^{i\boldsymbol{\alpha\sigma}}$ is block-diagonal as well. As exp is surjective for $SU(2)$, so we get all group elements this way, so that r is reducible which is a contradiction.

Problem 2: Let

$$P(\boldsymbol{z}) = \sum_{k=0}^{d} a_k z_1^k z_2^{d-k} \,. \tag{0.6}$$

be a complex homogeneous polynomial of degree d and call the complex $d+1$-dimensional vector space of all such polynomials (for a fixed d) Π_d. The map

$$r_d(g)P := P(g^{-1}\boldsymbol{z}) \,. \tag{0.7}$$

where $g^{-1} \in SU(2)$ acts on $\boldsymbol{z} = (z_1, z_2)$ as

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \to g^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{0.8}$$

defines a representation r_d of $SU(2)$ acting on $\mathbb{C}^{d+1} = \Pi_d$. Find the Lie algebra representation ρ_d associated with this representation.

solution:

To work this out, let's first make a strategy. We need to compute

$$\frac{\partial}{\partial t} r_d(g(t))\bigg|_{t=0} \tag{0.9}$$

for $g(t)$ a path in $SU(2)$. We can make the simple choice of looking at $e^{it\sigma_j}$. We didn't write down $r_d(g)$ as a matrix, so let's just see how it acts on a given basis vector. I.e. we can work this out for each j by looking at what happens to each monomial (= basis vector), so let's choose $z_1^k z_2^{d-k}$.

First consider

$$g(t) = e^{it\sigma_3} = e^{it\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}} \tag{0.10}$$

acting on $z_1^k z_2^{d-k}$. The Pauli matrix in the exponential maps (note the signs originating from using g^{-1})

$$\begin{aligned} z_1 &\to e^{-it} z_1 \\ z_2 &\to e^{it} z_2 \end{aligned} \tag{0.11}$$

Hence

$$\frac{\partial}{\partial t} r(g(t)) z_1^k z_2^{d-k}\bigg|_{t=0} = \frac{\partial}{\partial t} z_1^k z_2^{d-k} e^{it(d-2k)}\bigg|_{t=0} = i(d-2k) z_1^k z_2^{d-k}\,. \tag{0.12}$$

The Lie algebra representation $\rho_d(i\sigma_3)$ of $SU(2)$ rescales every basis vector by $i(d-2k)$.

Now we can work out a more complicated case.

$$g(t) = e^{it\sigma_1} = e^{it\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}} \tag{0.13}$$

acting on $z_1^k z_2^{d-k}$. Think of writing out this monstrosity as a power series in t. Anything quadratic or higher order in t for every factor will give zero at the end of the day no matter what it multiplies. Hence it is good enough to write out the action to linear order in t:

$$\begin{aligned} z_1 &\to z_1 - itz_2 \\ z_2 &\to z_2 - itz_1 \end{aligned} \tag{0.14}$$

So

$$\begin{aligned} \frac{\partial}{\partial t} r(g(t)) z_1^k z_2^{d-k}\bigg|_{t=0} &= \frac{\partial}{\partial t}(z_1 - itz_2)^k (z_2 - itz_1)^{d-k}\bigg|_{t=0} \\ &= -ikz_1^{k-1} z_2^{d-k+1} - i(d-k) z_1^{k+1} z_2^{d-k-1}\,. \end{aligned} \tag{0.15}$$

All we did here is to collect all the terms linear in t. The map $\rho(i\sigma_1)$ is

$$z_1^k z_2^{d-k} \to -ikz_1^{k-1} z_2^{d-k+1} - i(d-k) z_1^{k+1} z_2^{d-k-1} \tag{0.16}$$

You can work out $\rho(\sigma_2)$ analogously or by using

$$\rho(\sigma_2) = \rho(\tfrac{i}{2}[\sigma_1, \sigma_3]) = \tfrac{i}{2}[\rho(\sigma_1), \rho(\sigma_3)] \tag{0.17}$$