30. Show that any irreducible complex representation of $SO(3)$ also defines an irreducible complex representation of $SU(2)$.

    **solution:**

    Let us assume that we are given an irredicible representation $r_{SO(3)}$ of $SO(3)$, i.e.

    $$r_{SO(3)} : SO(3) \to GL(n, \mathbb{C}) \tag{0.1}$$

    is a homomorphism and there is no complex sub-vectorspace $W$ of $\mathbb{C}^n$ (except $\mathbb{C}^n$ and $\{0\}$) s.t.

    $$r_{SO(3)}(g)w \in W \ \forall w \in W, \forall g \in SO(3)\,. \tag{0.2}$$

    As shown in the lectures there is a close relationship between $SO(3)$ and $SU(2)$, i.e. there is a homomorphism $\pi : SU(2) \to SO(3)$. We can hence define the following composition

    $$r_{SU(2)} := r_{SO(3)} \circ \pi \tag{0.3}$$

    which takes any $h \in SU(2)$ to an element of $SO(3)$ and then to an element of $GL(n, \mathbb{C})$, so in effect we are taking any $h \in SU(2)$ to an element of $GL(n, \mathbb{C})$. As compositions of homomorphisms are again homomorphisms, this is a homomorphism as well and hence defines a representation of $SU(2)$.

    Now let's investigate irredicibility. As we have seen $\pi$ is surjective, i.e. we can write any $g \in SO(3)$ as $\pi(h)$ for some $h \in SU(2)$. As there is no complex sub-vectorspace $W$ of $\mathbb{C}^n$ (except $\mathbb{C}^n$ and $\{0\}$) s.t.

    $$r_{SO(3)}(g)w \in W \ \forall w \in W, \forall g \in SO(3)\,. \tag{0.4}$$

    and we can write any such $g$ as $g = \pi(h)$, it follows that there is no complex sub-vectorspace $W$ of $\mathbb{C}^n$ (except $\mathbb{C}^n$ and $\{0\}$) s.t.

    $$r_{SU(2)}(h)w \in W \ \forall w \in W, \forall h \in SU(2)\,. \tag{0.5}$$

    So $r_{SU(2)}$ is irredicible as well.

31. Let $V$ be the vector space of complex $2 \times 2$ matrices, and let $g \in SU(2)$ act on $A \in V$ as

    $$A \to gAg^\dagger\,.$$

    a) Show that this defines a representation $r$ of $SU(2)$.

    b) Show that $r$ is reducible.
    [hint: think about what happens to $\mathrm{tr}A$.]

c) Decompose $r$ into irreducible representations.

**solution:**

a) This we have already done may times. For all $g \in SU(2)$ this is a linear invertible map on $V$. Btw, for the inverse just observe that if

$$r(g) : A \to_g gAg^\dagger \tag{0.6}$$

then

$$r(g^{-1}) : gAg^\dagger \to= g^{-1}gAg^\dagger(g^{-1})^\dagger = A \tag{0.7}$$

using $(g^{-1})^\dagger = (g^\dagger)^{-1}$.

b) We need to find an invariant subspace to show this. As per the hint, lets investigate what happens to the trace of $A$:

$$\mathrm{tr}A \to \mathrm{tr}gAg^{-1} = \mathrm{tr}A\,. \tag{0.8}$$

Now what this implies is that we can never map matrices with a vanishing trace to ones with a non-vanishing trace. Let's try to understand this a bit more clearly and in terms of subspaces of $V$. The matrices $A$ have the form

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{0.9}$$

and we think of the four complex components $a_{ij}$ as components of a vector in $V$ (which is isomorphic to $\mathbb{C}^4$) that we chose to write as a matrix. Within this vector space there is a complex three-dimensional vector subspace $W$ defined by $a_{11} + a_{22} = 0$, and as (**??**) shows, the group action on $V$ maps vectors in $W$ again to vectors in $W$, i.e. $W$ is an invariant subspace. More concretely, $W$ is the subspace of matrices of the form

$$W = \left\{ A \,\middle|\, A = \begin{pmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{pmatrix}, (z_1, z_2, z_3) \in \mathbb{C}^3 \right\}. \tag{0.10}$$

This is indeed a vector subspace: the sum of any two such matrices and a scalar multiple looks again like this.

c) There are various ways to approach this. Note that the canonical inner form $|z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2$ on $\mathbb{C}^4$ is left invariant under the action of $SU(2)$. We can write this as

$$\begin{aligned} |z_1|^2 + |z_2|^2 + |z_3|^2 + |z_4|^2 = A_{ij}\bar{A}_{ij} &\to g_{ik}A_{kl}g^\dagger_{lj}\bar{g}_{im}\bar{A}_{mn}g^T_{nj} \\ = g^\dagger_{mi}g_{ik}g^\dagger_{lj}g_{jn}A_{kl}\bar{A}_{mn} &= \delta_{mk}\delta_{ln}A_{kl}\bar{A}_{mn} = A_{kl}\bar{A}_{kl}\,. \end{aligned} \tag{0.11}$$

This is hence a unitary representation and the orthogonal subspace to $W$ must be left invariant as well. This subspace $W^\perp$ is the one-dimensional subspace of $V$ containing matrices of the form

$$W^\perp = \left\{ A \middle| A = \begin{pmatrix} z_4 & 0 \\ 0 & z_4 \end{pmatrix}, z_4 \in \mathbb{C} \right\} . \tag{0.12}$$

which indeed form an invariant subspace under the group action as you can check easily.

For any $A$ we can write

$$A = \begin{pmatrix} z_1 & z_2 \\ z_3 & -z_1 \end{pmatrix} + \begin{pmatrix} z_4 & 0 \\ 0 & z_4 \end{pmatrix} , \tag{0.13}$$

which shows how to decompose any $A \in V$ under $V = W \oplus W^\perp$.

Hence $r$ decomposes into a one-dimensional and a three-dimensional complex representation. Are these irredicible? A complex one-dimensional representation is irredicible by definition, so there is nothing to do here. The other representation is likewise irreducible, this is just a complexified version of the adjoint representation which we know to be irredicible.

Have a nice holiday break!