1) Let $\mathbb{C}$ be the complex numbers and $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. Which of these is a group under addition? Which of these is a group under multiplication?

   Solution:
   $\mathbb{C}$ satisfies the definition of a group under addition, here the inverse of z is $-z$ and the neutral element is 0. Under multiplication, 0 does not have an inverse, so it is not a group under multiplication.

   $\mathbb{C}^*$ satisfies the definition of a group under multiplication, here the inverse of z is z^{-1} and the neutral element is 1. It is not a group under addition as it does not contain a neutral element for that.

2) Consider the set S of real $n \times n$ matrices.

   a) Show that S this is a (real) vector space V.
   b) Let $U \subset S$ be the set of matrices with determinant 1. Is U a vector space as well?
   c) For any matrix Q in V define a map

   $$g_M : Q \to M^{-1}QM$$

   where M is a fixed invertible matrix. Show that g_M is a linear map on V.

   Solution:

   a) We need to show that for any vectors $v, v' \in V$, $v + v'$ and av for $a \in \mathbb{R}$ is also in V. For arbitrary real $n \times n$ matrices S, S', both the sum $S + S'$ and a rescaled aS is another real $n \times n$ matrix. Hence this is a vector space.
   b) Here we are in trouble: is $S \in U$ we have $S = 1$. But to be a vector space, U also has to contain aS for all $a \in \mathbb{R}$. But $\det aS = a^n$, so aS is not in U. Hence U is not a vector space.
   c) First note that for $Q \in V$, $g_M(Q)$ is another real $n \times n$ matrix, the map hence maps elements in V to elements in V again. To check linearity we work out that for $Q, P \in V$ we have

   $$g_M(Q+P) = M^{-1}(Q+P)M = M^{-1}QM + M^{-1}PM = g_M(Q) + g_M(P) \tag{0.1}$$

   so g_M is a linear map acting on V.

3) a) By working out the derivative of

   $$\lim_{n\to\infty} (1 + i\phi/n)^n$$

with respect to ϕ, show that this expression satisfies the same differential equation as $e^{i\phi}$. You may assume that you can swap the order of the limit and taking the derivative.

As both functions have the same value at $\phi = 0$ this implies that they are equal by the uniqueness of solutions of ordinary differential equations.

b) Consider a square matrix A and let $g = e^{iA}$, which is defined via the Taylor series of the exponential. Show that

$$\lim_{n\to\infty} (\mathbb{1} + iA/n)^n = e^{iA} \,. \tag{0.2}$$

Solution:

a) The differential equation of the exponential $g = e^{i\phi}$ is

$$\frac{\partial}{\partial\phi} g = ig \,. \tag{0.3}$$

We now work out

$$\frac{\partial}{\partial\phi}(1 + i\phi/n)^n = n(1 + i\phi/n)^{n-1}\frac{i}{n} = \frac{i(1 + i\phi/n)^n}{1 + i\phi/n} \,. \tag{0.4}$$

In the limit $n \to \infty$, the rhs converges to $i(1 + i\phi/n)^n$. Assuming that differentiation and taking the limit commute (which we can do by using uniform convergence), we have hence shown that

$$\frac{\partial}{\partial\phi}\lim_{n\to\infty}(1 + i\phi/n)^n = i\lim_{n\to\infty}(1 + i\phi/n)^n \,. \tag{0.5}$$

Letting $e(\phi) \equiv \lim_{n\to\infty}(1 + i\phi/n)^n$, we have just shown that

$$\frac{\partial}{\partial\phi} e(\phi) = ie(\phi) \tag{0.6}$$

and can conclude that $e(\phi) = ce^{i\phi}$ using uniqueness of solutions of ordinary differential equations. Setting $\phi = 0$ the constant of proportionality c is seen to be 1 and we are done.

b) We can repeat the proof in the lecture by replacing ϕ with A in every step.