4 Show using index notation that

a) $(gh)^\dagger = h^\dagger g^\dagger$

b) $\mathrm{tr}(gh) = \mathrm{tr}(hg)$

c) $(g\boldsymbol{v}) \cdot (h\boldsymbol{w}) = \boldsymbol{v}\,(g^T h)\,\boldsymbol{w}$

d) $\det g^\dagger = \overline{\det g}$

where g and h are complex $n \times n$ matrices and $\boldsymbol{v}$ and $\boldsymbol{w}$ are n-dimensional vectors. **Solution:**

a) First note that

$$(gh)_{ij} = g_{ik} h_{kj} \tag{0.1}$$

where we are using summation convention, i.e. k is being summed over. All that the † does is transposition, i.e. swapping the two indices of a matrix, together with complex conjugation, so that

$$(gh)^\dagger_{ij} = \overline{g_{jk} h_{ki}} = \bar{g}_{jk} \bar{h}_{ki} = h^\dagger_{ik} g^\dagger_{kj} = (h^\dagger g^\dagger)_{ij} \tag{0.2}$$

b) We have $\mathrm{tr} gh = (gh)_{ii}$ so that

$$\mathrm{tr} gh = g_{ij} h_{ji} = h_{ji} g_{ij} = \mathrm{tr} hg \tag{0.3}$$

c) First recall that $(g\boldsymbol{v})_i = g_{ij} v_j$ and $\boldsymbol{a} \cdot \boldsymbol{b} = a_i b_i$, so that

$$(g\boldsymbol{v}) \cdot (h\boldsymbol{w}) = g_{ij} v_j h_{ik} w_k = v_j g^T_{ji} h_{ik} w_k = \boldsymbol{v}\, g^T h\, \boldsymbol{w} \tag{0.4}$$

d) We can write the determinant as

$$\det g = \epsilon_{i_1 i_2 \dots i_n}\, \epsilon_{j_1 j_2 \dots j_n}\; g_{i_1 j_1}\, g_{i_2 j_2} \cdots g_{i_n j_n} \tag{0.5}$$

using the ϵ tensor with n indices which I have called $i_1 \cdots i_n$ and $j_1 \cdots j_n$. Take a moment to unpack this and convince yourself that this produces the usual rule for e.g. a 2×2 matrix. Note that we are summing over all indices and it does not matter what we call them.

Now we are ready to work out

$$\begin{aligned} \det g^\dagger &= \epsilon_{i_1 i_2 \dots i_n}\, \epsilon_{j_1 j_2 \dots j_n}\; g^\dagger_{i_1 j_1}\, g^\dagger_{i_2 j_2} \cdots g^\dagger_{i_n j_n} \\ &= \epsilon_{i_1 i_2 \dots i_n}\, \epsilon_{j_1 j_2 \dots j_n}\; \bar{g}_{j_1 i_1}\, \bar{g}_{j_2 i_2} \cdots \bar{g}_{j_n i_n} \\ &= \epsilon_{j_1 j_2 \dots j_n}\, \epsilon_{i_1 i_2 \dots i_n}\; \bar{g}_{j_1 i_1}\, \bar{g}_{j_2 i_2} \cdots \bar{g}_{j_n i_n} \\ &= \overline{\epsilon_{j_1 j_2 \dots j_n}\, \epsilon_{i_1 i_2 \dots i_n}\; g_{j_1 i_1}\, g_{j_2 i_2} \cdots g_{j_n i_n}} = \overline{\det g} \end{aligned} \tag{0.6}$$

5. For a general $k \times k$ matrix M show that

   (a) $\det e^M = e^{\mathrm{tr} M}$.

   (b) Use this to conclude that for $g = e^M$ we have $\log \det g = \mathrm{tr} \log g$. Here the log of a matrix is defined as the inverse function of the exponential.

   Solution:

   (a) We can start as in the lecture to find

   $$\det e^M = \lim_{n \to \infty} [\det(\mathbb{1} + M/n)]^n \tag{0.7}$$

   Now consider $\det(\mathbb{1} + M/n)$. The terms appearing in the determinant are all products of entries of the matrix $\mathbb{1} + M/n$. Terms that do not contain diagonal elements have a factor n^{-k}, whereas each diagonal factor in a summand of the determinant replaces one factor of n^{-1} by one factor $(1 + M_{ii}/n)$ (where i is the index of the diagonal element). The leading terms in the limit $n \to \infty$ are hence coming from the term in the determinant which has only factors from the diagonal of $\mathbb{1} + M/n$:

   $$\lim_{n \to \infty} [\det(\mathbb{1} + M/n)]^n = \lim_{n \to \infty} \left[\prod_i \left(1 + \frac{M_{ii}}{n} \right) \right]^n \tag{0.8}$$

   A similar feature appears when expanding the product, the leading terms are those that contain only a single factor of n^{-1}:

   $$\prod_i \left(1 + \frac{M_{ii}}{n} \right) = 1 + \frac{1}{n} \sum_i M_{ii} + \mathcal{O}(n^{-2}) \tag{0.9}$$

   Hence

   $$\det e^M = \lim_{n \to \infty} [\det(\mathbb{1} + M/n)]^n = \lim_{n \to \infty} \left[1 + \frac{\mathrm{tr} M}{n} \right]^n = e^{\mathrm{tr} M} \tag{0.10}$$

   (b) We have $g = e^M$ and $M = \log g$. Applying the above formula we find

   $$\log \det g = \log \det e^M = \log e^{\mathrm{tr} M} = \mathrm{tr} M = \mathrm{tr} \log g \, . \tag{0.11}$$

   In the physics literature, the relationship proven in a) is mostly stated in the above form as 'log det = tr log'.

6. Show that the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.12}$$

satisfy

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \,. \tag{0.13}$$

Solution:

We work out

$$\begin{aligned} [\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_3 = 2i\epsilon_{123}\sigma_3 \end{aligned} \tag{0.14}$$

$$\begin{aligned} [\sigma_2, \sigma_3] &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 2i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2i\sigma_1 = 2i\epsilon_{231}\sigma_1 \end{aligned} \tag{0.15}$$

$$\begin{aligned} [\sigma_3, \sigma_1] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2i\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2i\sigma_2 = 2i\epsilon_{312}\sigma_2 \end{aligned} \tag{0.16}$$

The remaining cases of commutators follow from the antisymmetry of both the commutator and the ϵ_{ijk}.