10. Let G be the set of complex 2×2 matrices of the form

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

for $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 \neq 0$.

a) Show that G is a group using matrix multiplication as the group operation.

b) Show that $SU(2)$ is a subgroup of G.

c) Show that $V := \{\gamma | g = e^{i\gamma} \in G\}$ is a vector space and find a basis for V.

solution:

a) We can do this in a straight-forward way by checking the group properties in this explicit form. A little more elegant is to realize that these are exactly the complex 2×2 matrices that obey

$$g^\dagger = g^{-1} \ \det(g) \tag{0.1}$$

with $\det g \neq 0$. Writing

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{0.2}$$

this implies that

$$g^\dagger = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{0.3}$$

which results in the general form above.

Now this is clearly obeyed by the identity, if g obeys it then

$$(g^{-1})^\dagger = (g^\dagger)^\dagger \det g = g \det g^{-1} \tag{0.4}$$

so the inverse is in G as well. Matrix multiplication is associative and finally

$$(gh)^\dagger = h^\dagger g^\dagger = h^{-1} \det h \ \ g^{-1} \det g = (gh)^{-1} \det gh \tag{0.5}$$

so that composition of group elements makes new group elements.

Remark: this is nothing but the group of quaternions written as complex matrices.

b) $SU(2)$ are those $g \in G$ with $\det g = |\alpha|^2 + |\beta|^2 = 1$. This feature is preserved when taking the inverse or multiplying two elements of $SU(2)$, so that $SU(2)$ is a subgroup.

c) There are two ways of approaching this. Let me first use part c), which immediately tells me that I can use $i\sigma_j$ with σ_j the Pauli matrices in the exponential. We can write any $g \in G$ that is also in $SU(2)$ as

$$g_{SU(2)} = \exp \sum_j i a_j \sigma_j \, . \tag{0.6}$$

Now this is all of it for $SU(2)$ which is real 3-dimensional (there are three real a_j), but how about the present case? Any element of G is determined by fixing the complex numbers α and β s.t. $|\alpha|^2 + |\beta|^2 \neq 0$ and this is four real parameters. We are hence looking for one more direction. What do matrices in G look like that are not in $SU(2)$? Here is an example: for any $\alpha = e^r$ with $r \neq 0$ we are not in $SU(2)$:

$$g = \begin{pmatrix} e^r & 0 \\ 0 & e^r \end{pmatrix} = \exp r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, . \tag{0.7}$$

Now we can try writing a $g \in G$ as

$$g = \exp\left(\sum_j i a_j \sigma_j\right) \exp\left(r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \exp\left(\sum_j i a_j \sigma_j + r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{0.8}$$

as the identity matrix commutes with everything. We hence arrive at the set of all γs as

$$\left\{ \sum_j a_j \sigma_j - ir \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \middle| \, a_j \in \mathbb{R}, r \in \mathbb{R} \right\} \tag{0.9}$$

We are free to choose the a_j and r in the real numbers so that the set of all γ just described is $\mathbb{R}^4$, which is a vector space. Equally, you can show that addition and scalar multiplication preserves this set. A basis is given by $\sigma_j, j = 1, 2, 3$ and i times the identity matrix.

What is slightly unsatisfactory about this is that we don't know if we might have missed something, i.e. if the above is really V. What the above argument shows is that the general g we have constructed is a product of something in $SU(2)$ with the identity matrix times a positive number, so we can reach

$$e^{i\gamma} = g_{SU(2)} \begin{pmatrix} e^r & 0 \\ 0 & e^r \end{pmatrix} = e^r g_{SU(2)} \tag{0.10}$$

We can simply rescale any element in G by a positive number to reach an element in $SU(2)$, so the above is in fact general and we are done.

A faster way is to realize that

$$g^\dagger = (e^{i\gamma})^\dagger = e^{-i\gamma^\dagger} = g^{-1}\det g = e^{-i\gamma}e^{i\,\mathrm{tr}\gamma} \tag{0.11}$$

implies

$$\gamma^\dagger = \gamma - \mathbb{1}\mathrm{tr}\gamma \tag{0.12}$$

which forms a vector space: we have

$$(c\gamma)^\dagger = c\gamma - \mathbb{1}\mathrm{tr}c\gamma \tag{0.13}$$

for $c \in \mathbb{R}$ and

$$(\delta+\gamma)^\dagger = \delta^\dagger + \gamma^\dagger = \delta + \gamma - \mathbb{1}\left(\mathrm{tr}\gamma + \mathrm{tr}\delta\right) = \delta + \gamma - \mathbb{1}\mathrm{tr}\left(\gamma + \delta\right) \tag{0.14}$$

A basis of the vector space of solutions to $\gamma^\dagger = \gamma - \mathbb{1}\mathrm{tr}\gamma$ are the matrices we have found above, $\sigma_j, j = 1,2,3$ and $i\mathbb{1}$.

11. Which of the following sets are closed? Which are open? For all cases use the standard topology of $\mathbb{R}^n$ or a topology induced from it.

    (a) $\{0 < x < \pi\} \subset \mathbb{R}$ with coordinate x
    (b) $\{x_1 < -2\} \subset \mathbb{R}^2$ with coordinates (x_1, x_2)
    (c) $\{0 < x \leq \pi\} \subset \mathbb{R}$
    (d) $\{0 < x_1 < 1\} \subset \mathbb{R}^2$ with coordinates (x_1, x_2)
    (e) $\mathbb{R}^n \subseteq \mathbb{R}^n$
    (f) $\{(x_1, x_2) \subset \mathbb{R}^2 \,|\, x_1^2 \leq 42 - x_2^2\} \subset \mathbb{R}^2$ with coordinates (x_1, x_2)
    (g) $\{(x_1, x_2) | x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^3$
    (h) $\{(x_1, x_2) | x_1^2 + x_2^2 = 1\} \subset \{(x_1, x_2, x_3) | x_1^2 + x_2^2 + x_3^4 = 1\}$ with the topology induced from $\mathbb{R}^3$

    solution:

    (a) open
    (b) open
    (c) not open and not closed
    (d) open
    (e) open and closed
    (f) this is a closed disc of radius $\sqrt{42}$
    (g) closed

(h) closed: imposing only the second equation give a tube of varying radius along x_3, the first then gives a circle sitting inside of it.

12 Prove that arbitrary unions and finite intersections of open sets in $\mathbb{R}^n$ are again open. Why is the intersection of an infinite number of open sets not open in general ?

solution:

Let $U = \bigcup_{u \in S} u$ be the union of an infinite set S of open sets u. Let $\boldsymbol{p}$ be any point in U. Then it must be contained in one of the u and hence there is an open ball entirely contained in u because u is open. As U is the union of all of these, this ball is also contained in U.

For the second statement, let us start by considering a non-empty intersection between two open sets $U = U_1 \cap U_2$. For any point $\boldsymbol{p}$ in this intersection we can find a ball $B_{r_1}(\boldsymbol{p})$ centered at $\boldsymbol{p}$ that sits entirely in U_1, and a ball $B_{r_2}(\boldsymbol{p})$ that is entirely in U_2. Without loss of generality we can assume that $r_1 \leq r_2$, But this means that $B_{r_1}(\boldsymbol{p}) \subseteq B_{r_2}(\boldsymbol{p})$ so that $B_{r_1}(\boldsymbol{p}) \subset U$.

Now let $U = \bigcap_{u \in S} u$ for a finite set S. Consider any point $\boldsymbol{p} \in U$. By repeating the above argument a finite number of times, we will find a finite sized open ball sitting in U.

The latter argument fails for an arbitrary intersection $U_i, i \in \mathbb{N}$. Here, it can happen that the sizes r_i approach zero as $i \to \infty$. Letting $r_i \to 0$ for $i \to \infty$, the infinite intersection of open sets

$$\bigcap_{i=1}^{\infty} B_{r_i}(\boldsymbol{p}) = \boldsymbol{p}$$

is just a point, which is not an open set. Note that each r_i is finite, so each U_i is open.

13 Consider the sets of points in $\mathbb{R}^2$ with coordinates (x, y) defined implicitely by the following relations

a) $y = x^3$

b) $xy = 0$

c) $x^2 + y^4 = 1$

d) $x > y$

e) $y^2 + x^3 - 3x - 2 = 0$

Using the induced topology from $\mathbb{R}^2$, decide in each case if this is a differentiable manifold.

[hint: plot them! Note that the word 'differentiable' here refers to the manifold and not the functions I used to define a manifold. The two notions are not unrelated however, details are explained in the non-examinable example 1.10. but this is not needed to answer this question.]

solution:

a) This can be mapped to $\mathbb{R}$ using simply x as the coordinate, so this is in fact homeomorphic to $\mathbb{R}$ and it a manifold.

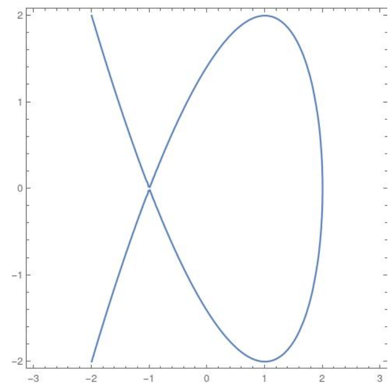
b) This is the union of two lines $x = 0$ and $y = 0$ meeting at the origin and is not a manifold. Using the topology induced from $\mathbb{R}^2$, there is no issue to define coordinates away from the point $(y, x) = (0, 0)$, we just cut out a little branch and map it to an open set in $\mathbb{R}$. However any open set U containing the point $(y, x) = (0, 0)$ also contains (a small piece at least from) both branches. Hence these open sets look like a cross, which is radically different from any open subset of $\mathbb{R}$. There cannot be any homeomorphism to an open subset of $\mathbb{R}$ for such a U.

   We can make a slightly more detailed argument about why that is as follows: choose a point p_a on the line $x = 0$, and an open interval on $xy = 0$ which connects it to $(0, 0)$, and then to a second point p_b on the line $x = 0$ beyond $(0, 0)$. Using that we want a continuous map to $\mathbb{R}$, this interval must be mapped to an open interval in $\mathbb{R}$ and $(0, 0)$ goes to $0 \in \mathbb{R}$ (say). The image of the interval on one branch gives us an open interval in $\mathbb{R}$. Its inverse image must be an open set as well, as we need our coordinate map to be a homeomorphism. The open sets containing $(0, 0)$ all contain points on the other branch as well, so it needs to be mapped to our interval $\subset \mathbb{R}$ as well. But this cannot be as we need a 1-1 map. Note that this problem disappears as soon as you either drop that our map and its inverse are continuous, or that it is 1-1.

c) This just looks like a dented circle and is a manifold.

d) This has dimension two, but is a manifold; we can just use the coordinates of $\mathbb{R}^2$ used in its description.

e) Let me call this set E. Plotting E reveals it looks like this



E is an example of what is commonly called an 'elliptic curve'. As can be seen from the plot, two branches cross in the point $(y, x) = (0, -1)$. This can be seen from the structure of the equation as well. For every x there are two values of y, except when

$$y^2 = -(x^3 - 3x - 2) = (2 - x)(1 + x)^2 = 0\,. \tag{0.15}$$

Note that double root at $x + 1 = 0$. We can write the above as

$$y = \pm(1 + x)\sqrt{2 - x}\,. \tag{0.16}$$

so that there are two branches which meet at $x = -1$.

Zooming in on this point, it looks the same as the example of $xy = 0$ considered above, so that this cannot be a manifold for the same reasons.

Here are some things you should discuss with your friends:

1. What is a topology, what is a topological space?
2. What is a manifold?
3. Why do we need to declare which sets we consider open (create a 'topological space') before we can define coordinate patches?