14. Consider the stereographic projection of the three-sphere S^3.

   a) Show that the inverse of the map between $\boldsymbol{\varphi}_\pm$ and the x_i is given by

$$x_1 = \frac{\pm(1-|\boldsymbol{\varphi}_\pm|^2)}{|\boldsymbol{\varphi}_\pm|^2+1}$$
$$\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} = \frac{2}{|\boldsymbol{\varphi}_\pm|^2+1}\boldsymbol{\varphi}_\pm$$

   b) Consider the coordinate patches defined by stereographic projection on S^3 and find the coordinate change from U_+ to U_-.

solution:

(a) We only give details for $\boldsymbol{\varphi}_-$, the computation for $\boldsymbol{\varphi}_+$ is analogous. The crucial part is finding x_1 as a function of $\boldsymbol{\varphi}_-$. First work out

$$|\boldsymbol{\varphi}_-|^2 = \frac{x_2^2+x_3^2+x_4^2}{(1-x_1)^2} = \frac{1-x_1^2}{(1-x_1)^2} = \frac{1+x_1}{1-x_1}$$

Now consider

$$|\boldsymbol{\varphi}_-|^2+1 = \frac{1+x_1+1-x_1}{1-x_1} = \frac{2}{1-x_1}$$

which we can use to find

$$x_1 = \frac{|\boldsymbol{\varphi}_-|^2-1}{|\boldsymbol{\varphi}_-|^2+1}\,.$$

This is x_1 as a function of φ_-. The other coordinates are now $\boldsymbol{\varphi}_-$ times

$$1-x_1 = 1-\frac{|\boldsymbol{\varphi}_-|^2-1}{|\boldsymbol{\varphi}_-|^2+1} = \frac{|\boldsymbol{\varphi}_-|^2+1-(|\boldsymbol{\varphi}_-|^2-1)}{|\boldsymbol{\varphi}_-|^2+1} = \frac{2}{|\boldsymbol{\varphi}_-|^2+1} \tag{0.1}$$

(b) We are interested in $\varphi_-\varphi_+^{-1}$, which we can write as

$$\varphi_-(\boldsymbol{x}(\boldsymbol{\varphi}_+)) \tag{0.2}$$

We have

$$\begin{aligned} x_1 &= \frac{(1-|\boldsymbol{\varphi}_+|^2)}{|\boldsymbol{\varphi}_+|^2+1} \\ \begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} &= \frac{2}{|\boldsymbol{\varphi}_+|^2+1}\boldsymbol{\varphi}_+ \end{aligned} \tag{0.3}$$

and

$$\boldsymbol{\varphi}_-(\boldsymbol{x}) = \frac{1}{1-x_1}\begin{pmatrix} x_2 \\ x_3 \\ x_4 \end{pmatrix} . \tag{0.4}$$

so we get

$$\boldsymbol{\varphi}_- = \frac{1}{1-\frac{1-|\boldsymbol{\varphi}_+|^2}{|\boldsymbol{\varphi}_+|^2+1}} \frac{2\boldsymbol{\varphi}_+}{|\boldsymbol{\varphi}_+|^2+1} = \boldsymbol{\varphi}_+ \frac{2}{|\boldsymbol{\varphi}_+|^2+1} \frac{|\boldsymbol{\varphi}_+|^2+1}{|\boldsymbol{\varphi}_+|^2+1-1+|\boldsymbol{\varphi}_+|^2} = \boldsymbol{\varphi}_+/|\boldsymbol{\varphi}_+|^2 \tag{0.5}$$

As long as $|\boldsymbol{\varphi}_+|^2 \neq 0$, which is true in $U_+ \cap U_-$, this is a smooth function.

16. $O(1,1)$ are the real 2×2 matrices O which leave the bilinear form $x_1^2 - x_2^2$ invariant when acting on $\boldsymbol{x} = (x_1, x_2)$ as

$$\boldsymbol{x} \to O\boldsymbol{x} .$$

a) Show that $O(1,1)$ is a group using matrix multiplication.
b) Find the general form of elements of $O(1,1)$.
c) Give $O(1,1)$ the structure of a differentiable manifold by equipping it with a suitable topology and write down coordinate charts.
d) Find the tangent space of $O(1,1)$ at the identity element.

solution:
Acting with $O \in O(1,1)$ on $\mathbb{R}^2$ is supposed to leave the bilinear form $x_1^2 - x_2^2$ invariant. Imagine you have found two matrics O, O' with this property. We can act first with O

$$\boldsymbol{x} \to O\boldsymbol{x} \tag{0.6}$$

which leaves $x_1^2 - x_2^2$ invariant, after which we can then act with O' which again leaves $x_1^2 - x_2^2$ invariant. In summary, we have then acted with $O'O$, which makes using matrix multiplication as the group composition a good idea.

(a) Let us first find a condition that must be satisfied by matrices in $O(1,1)$. We need

$$x_1^2 - x_2^2 = \boldsymbol{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{x} \tag{0.7}$$

to stay invariant. This is mapped to

$$\boldsymbol{x}^T O^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} O\boldsymbol{x} \stackrel{!}{=} \boldsymbol{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{x} \tag{0.8}$$

so we find the condition

$$O^T L O = L \tag{0.9}$$

with $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Let us now check the group property using matrix multipliction as $\circ$. The condition above is solved by O the identity matrix, so we have the identity element. This equation also implies that $\det O \neq 0$, so an inverse exists. We can even write it down: $O^{-1} = LO^T L$ because $LO^T LO = L^2 = \mathbb{1}$. Now multiplying $O^T LO = L$ by O^{-1} from the right and $(O^T)^{-1}$ from the left shows

$$L = (O^T)^{-1} L O^{-1} = (O^{-1})^T L O^{-1} . \tag{0.10}$$

so the inverse satisfies the same equation. We have used that $(O^T)^{-1} = (O^{-1})^T$ which can be seen by

$$(O^T)^{-1} O^T = \mathbb{1} = (OO^{-1})^T = (O^{-1})^T O^T . \tag{0.11}$$

Finally we can observe that for $O'' = OO'$ with $O, O' \in O(1,1)$ we have

$$(O'')^T L O'' = (O')^T O^T L O O' = (O')^T L O' = L . \tag{0.12}$$

so $O'' \in O(1,1)$ as well.

(b) Note that $\det O = \pm 1$ by taking the determinant of $O^T LO = L$ on both sides. We can parametrize O as

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad O^{-1} = \pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{0.13}$$

and plug this into $O^T LO = L$ to find

$$a = \pm d \, , \qquad b = \pm c . \tag{0.14}$$

The determinant of O changes continuously along any path in O, which implies that there are hence (at least) two components, one for $\det O = 1$ and one for $\det O = -1$.

<u>$\det O = 1$</u> This implies that

$$O = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \tag{0.15}$$

with $a^2 - b^2 = 1$. We can parametrize solutions of this as

$$a = \pm \cosh \phi \qquad b = \sinh \phi . \tag{0.16}$$

There are two inequivalent solutions as $\cosh(\phi) > 0$, as $-\sinh(\phi) = \sinh(-\phi)$ the choice of sign of b does not make a difference. Note that the two types of solution we have found here are disjoint, a is always positive for one and always negative for the other. Hence we have found two disconnected components $O_+^\uparrow$ ($\det O > 0, a > 0$) and $O_+^\downarrow$ ($\det O > 0, a < 0$).

$\underline{\det O = -1}$ This implies that

$$O = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \tag{0.17}$$

with $-a^2 + b^2 = -1$.
Repeating the same steps as above we find $a = \pm\cosh(\phi)$ and $b = \sinh(\phi)$ for this case. Hence we get two more components $O_-^\uparrow$ ($\det O < 0, a > 0$) and $O_-^\downarrow$ ($\det O < 0, a < 0$).

In summary we have hence found 4 disjoint components. Only $O_+^\uparrow$ is a subgroup as it is the only component that contains the identity element.

(c) Note that we can map any component to a copy of $\mathbb{R}$ by the parametrization by ϕ in a one-to-one fashion. Let us call these maps $\Phi_\pm^{\uparrow\downarrow}$. These give us good coordinate charts if we simply use the topology in which these are continuous maps, i.e. we can simply declare that for any component of $O(1,1)$, a subset is open if the corresponding subset is open in $\mathbb{R}$.

(d) The component connected to $\mathbb{1}$ is $O_+^\uparrow$ and we can describe a path through $\mathbb{1}$ by

$$O(t) = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \tag{0.18}$$

for $t = -e..e$ with some $e > 0$. We work out

$$\frac{\partial}{\partial t} O(t)|_{t=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{0.19}$$

so that the tangent space at $\mathbb{1}$ is the vector space of matrices

$$T_{\mathbb{1}} = \left\{ \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} |v \in \mathbb{R} \right\} . \tag{0.20}$$

Note that

$$\exp\left[\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}\right] = \begin{pmatrix} \cosh(v) & \sinh(v) \\ \sinh(v) & \cosh(v) \end{pmatrix} \tag{0.21}$$