17. Show that for every $g \in GL(n,\mathbb{R}) \setminus O(n)$, i.e. $g \in GL(n,\mathbb{R})$ such that $g^T g \neq \mathbb{1}$, there is an open set U_g containing g such that U_g is entirely contained in $GL(n,\mathbb{R}) \setminus O(n)$.

    hint: $GL(n,\mathbb{R})$ inherits its topology from the vector space $V_{n\times n}$ of real $n\times n$ matrices, which is isomorphic to $\mathbb{R}^{n^2}$: the n^2 entries of such a matrix are just the components of a vector in $\mathbb{R}^{n^2}$ from this perspective. We can hence describe the open ball of radius r around a matrix M with components M_{ij} as

$$B_r(M) = \left\{ N \in V_{n\times n} | \sum_{ij} (N_{ij} - M_{ij})^2 < r \right\} . \tag{0.1}$$

Solution:

Consider a $g \in GL(n,\mathbb{R}) \setminus O(n)$. As $g^T g \neq \mathbb{1}$ we can write

$$g^T g = \mathbb{1} + C \tag{0.2}$$

for some matrix C. Furthermore, let $\det g = c \neq 0$.

Now consider an open ball of radius ϵ in $V_{n\times n} = \mathbb{R}^{n^2}$ centered at g. We can describe every g_b in this ball as

$$g_b = g + \epsilon\Delta \tag{0.3}$$

for a number ϵ and a matrix Δ with $\sum_{ij} \Delta_{ij}^2 = 1$. We have

$$g_b^T g_b = \mathbb{1} + C + \epsilon(\Delta^T g + g^T \Delta) + \epsilon^2 \Delta^T \Delta\,. \tag{0.4}$$

We want to show that we can choose ϵ such that there is no g_b in $B_\epsilon(g)$ with $g_b^T g_b = \mathbb{1}$, i.e. there is no g_b in $B_\epsilon(g)$ with $g_b \in O(n)$. This can only happen if we can find Δ and ϵ such that

$$C + \epsilon(\Delta^T g + g^T \Delta) + \epsilon^2 \Delta^T \Delta = 0 \tag{0.5}$$

We can assume that $C_{ij} \neq 0$ for some i,j (otherwise $g^T g = \mathbb{1}$ and so $g \in O(n)$). Now in the above equation g is fixed, C is fixed (it depends on g) and the components of Δ are bounded. By choosing ϵ small enough, we can hence make sure that

$$C_{ij} > \epsilon \left(\Delta^T g + g^T \Delta\right)_{ij} + \epsilon^2 \left(\Delta^T \Delta\right)_{ij} = 0 \tag{0.6}$$

for all Δ. But this means that none of the g_b are in $O(n)$ for such an ϵ, i.e. they are all in $V_{n\times n} \setminus O(n)$ as this works for any $g \in V_{n\times n} \setminus O(n)$. We have hence shown that $V_{n\times n} \setminus O(n)$ is open.

What we were actually interested in is not showing that $V_{n\times n} \setminus O(n)$ is open but that $GL(n,\mathbb{R}) \setminus O(n)$ is open. We are in $GL(n,\mathbb{R})$ if $\det g \neq 0$. We can now repeat a similar argument as above to show that if $\det g = c$, then of course c is some finite number and for a small enough ϵ, all matrices in $B_\epsilon(g)$ also have $\det g_b \neq 0$. Note that the determinant is just a polynomial of the entries of g_b. But this means that we can choose ϵ small enough such that $B_\epsilon(g)$ is entirely in $GL(n,\mathbb{R})$. Hence such $B_\epsilon(g)$ are open in $GL(n,\mathbb{R})$.

In summary, if $g_b \in GL(n,\mathbb{R}) \setminus O(n)$ we can choose ϵ small enough such that for all $g_b \in B_\epsilon(g)$ we have $\det g_b \neq 0$ and $g_b^T g_b \neq \mathbb{1}$. Hence there is an open set U_g of $GL(n,\mathbb{R})$ for any such g that is entirely in $GL(n,\mathbb{R}) \setminus O(n)$.

18. $GL(n,\mathbb{C})$ is the group of invertible complex $n \times n$ matrices. Show that $GL(n,\mathbb{C})$ is a Lie group.

Solution:

Using Theorem 1.3. from the lectures, we can do this by showing that $GL(n,\mathbb{C})$ is a closed subgroup of $GL(m,\mathbb{R})$ for some m.

Let us first rewrite complex multiplication as real matrix multiplication. For $z \in \mathbb{C}$ write this as a vector $z_R := (\zeta_+, \zeta_-) \in \mathbb{R}^2$ by setting $z = \zeta_+ + i\zeta_-$. As

$$az = a_+\zeta_+ - a_-\zeta_- + i(a_+\zeta_- + a_-\zeta_+) \tag{0.7}$$

we can write multiplication of z by a as multiplying a vector z_R in $\mathbb{R}^2$ by a 2×2 matrix a_R

$$a_R z_R = \begin{pmatrix} a+ & -a_- \\ a_- & a_+ \end{pmatrix} \begin{pmatrix} \zeta_+ \\ \zeta_- \end{pmatrix} \tag{0.8}$$

This implies that for a complex matrix A acting on a complex vector $\boldsymbol{z}$ with n elements $z_i = \zeta_{i+} + i\zeta_{i-}$ we can write this as a real $2n \times 2n$ matrix acting on a real vector with $2n$ components:

$$A\boldsymbol{z} \leftrightarrow A_R \boldsymbol{z}_R \tag{0.9}$$

where each entry $a_{ij} = a_{ij+} + ia_{ij-}$ in A is replaced by the matrix

$$a_{ij} = \begin{pmatrix} a_{ij+} & -a_{ij-} \\ a_{ij-} & a_{ij+} \end{pmatrix} \tag{0.10}$$

and

$$z_R = (\zeta_{1+}, \zeta_{1-}, \zeta_{2+}, \zeta_{2-}, \cdots \zeta_{n+}, \zeta_{n-}) \tag{0.11}$$

We can hence translate every element of $GL(n,\mathbb{C})$ into a real $2n \times 2n$ matrix. This matrix is in $GL(n,\mathbb{R})$: as the kernel of A is trivial there is no $\boldsymbol{z} \neq 0$

such that $A\boldsymbol{z} = 0$. Writing this in purely real terms implies that there is no $\boldsymbol{z}_R \neq 0$ such that $A_R \boldsymbol{z}_R = 0$. Clearly this also satisfies the conditions of being a subgroup.

Finally, we want to see that it is a closed subgroup. Not all elements in $GL(2n, \mathbb{R})$ are of the form that A_R takes. Restricting the entries of a general matrix such that it has the form of A_R amounts to choosing a closed subspace: we set to zero all entries in a general $GL(n, \mathbb{R})$ matrix that depart from the structure found above. A detailed argument for this would go along the same lines as the solution to problem 17.

Alternatively, one can go through the same steps that we used in the lectures to show that $GL(n, \mathbb{R})$ is a Lie group and show that the group composition and inverse are differentiable maps.

19. Find the dimension of the group $SO(n)$ by finding the dimension of its Lie algebra.

solution: As dicussed in the lecture, the dimension of the group is the same as that of its Lie algebra. By repeating the same argument done for $SO(3)$, we find that writing an element of $SO(n)$ as an exponential of a matrix γ, it must be that $\gamma^T = -\gamma$. The exponential map is surjective, so we can in fact write any group element like this. Hence the dimension of the group is equal to the dimension of the vector space of real matrices that obey $\gamma^T = -\gamma$. Notice that this implies that the diagonal elements of γ are zero and that γ is uniquely specified by fixing its elements above the diagonal to be arbitrary real numbers. There are

$$(n^2 - n)/2 = \frac{n(n-1)}{2} \tag{0.12}$$

such elements, which is hence the (vector space) dimension of the Lie algebra $\mathfrak{so}(n)$ of $SO(n)$ and also the (manifold) dimension of $SO(n)$

20. Consider the set G of matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{R}, ac \neq 0 \right\} \tag{0.13}$$

(a) Show that G is a Lie group using matrix multiplication as the group composition.

(b) Find the Lie algebra $\mathfrak{g}$ of G.

(c) Compute the exponentials of the basis elements of the Lie algebra you have found.

solution:

(a) We compute

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix} \tag{0.14}$$

so the product of any two of these is also in G. Also we can check that

$$g^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \tag{0.15}$$

is in G for any $g \in G$.

It is furthermore a subgroup of $GL(2,\mathbb{R})$ that is closed, so that it must be a Lie group. Here are more details: writing a general matrix in $GL(2,\mathbb{R})$ as

$$\begin{pmatrix} a & b \\ d & c \end{pmatrix} \tag{0.16}$$

with $ac - bd \neq 0$, we can characterize G by $d = 0$. This is a closed condition, for any $d \neq= 0$ we can find a little open ball for which still $d \neq 0$.

(b) We need define an appropriate number of paths in G. As G is real three-dimensional, the tangent space at $\mathbb{1}$ is a three-dimensional vector space. We can write every element of G as

$$g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} e^x & y \\ 0 & e^z \end{pmatrix} \tag{0.17}$$

as $a \neq 0$ and $c \neq 0$. Here $(x, y, z) \in \mathbb{R}^3$. We can hence write some paths in G as

$$\begin{pmatrix} e^{xt} & yt \\ 0 & e^{zt} \end{pmatrix} \tag{0.18}$$

which all go through $g = \mathbb{1}$ at $t = 0$. Other ways of writing paths are fine too of course, but the above is the most convenient. Now we can work out

$$\frac{\partial}{\partial t}\begin{pmatrix} e^{xt} & yt \\ 0 & e^{zt} \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} . \tag{0.19}$$

The set of these matrices is isomorphic to $\mathbb{R}^3$ as $(x, y, z) \in \mathbb{R}^3$. As we already know that G is real three-dimensional this spans the whole Lie algebra of G. Note that these might look the same as the elements of G but, now x, y are allowed to vanish (after all, this is a vector space which must contain the zero vector).

(c) We can work out

$$\exp\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ 0 & 1 \end{pmatrix} \tag{0.20}$$

$$\exp\begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^z \end{pmatrix} \tag{0.21}$$

$$\exp\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \tag{0.22}$$

Here are some things you should discuss with your friends:

1. What is a Lie group?
2. How can you find the Lie algebra of a matrix Lie group?