21. Writing a vector $(v_1, v_2, v_3) \in \mathbb{R}^3$ as

$$M_{\boldsymbol{v}} = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} .$$

consider the action of $g \in SU(2)$ on $\mathbb{R}^3$ defined by

$$F(g) : M_{\boldsymbol{v}} \mapsto g M_{\boldsymbol{v}} g^\dagger .$$

Show that this is a representation, and that this representation is the adjoint representation of $SU(2)$.

22. Let $\mathbf{q} \in \mathbb{C}^n$ be acted on in the defining (also called 'fundamental') representation of $SU(n)$ and γ in the adjoint representation of $SU(n)$ (this is often expressed as $\mathbf{q}$ 'lives' in the fundamental and γ 'lives' in the adjoint of $SU(n)$.)

Describe the action of $SU(n)$ on

i) $\mathbf{v} = \gamma \mathbf{q}$

ii) $\bar{\mathbf{q}}$

iii) A matrix Q with components $Q_{ij} = q_i q_j$

and decide in each case if this defines a representation.

23. Let $g \in SO(3)$ be given by

$$g = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Find the action of g in the adjoint representation and describe it using a basis of the vector space $\mathfrak{so}(3)$. As $\mathfrak{so}(3)$ is the same as $\mathbb{R}^3$, we can describe its elements as column vectors after having chosen a basis. Using the basis you have chosen, write the adjoint action as a 3×3 matrix acting on a column vector.

24. Let G be a Lie group and H be a subgroup of G that is also a Lie group.

a) Explain why any representation $r(G)$ of G also gives us a representation $r(H)$ of H.

b) Let's assume $r(G)$ is irreducible. Can you think of an example where the representation $r(H)$ is reducible? Can you think of an example where the representation $r(H)$ is irreducible?

Here are some things you should discuss with your friends:

1. What is the geometric meaning of the Lie algebra of a Lie group?
2. When and how can you recover a Lie group from its Lie algebra?
3. What is a group representation?