21. Writing a vector $(v_1, v_2, v_3) \in \mathbb{R}^3$ as

$$M_{\boldsymbol{v}} = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} .$$

consider the action of $g \in SU(2)$ on $\mathbb{R}^3$ defined by

$$F(g) : M_{\boldsymbol{v}} \mapsto gM_{\boldsymbol{v}}g^\dagger .$$

Show that this is a representation, and that this representation is the adjoint representation of $SU(2)$.

solution:

Clearly, the set of matrices above forms a vector space which is isomorphic to $\mathbb{R}^3$. Choosing the Pauli matrices as a basis, we can write

$$M_{\boldsymbol{v}} = \sum_j v_j \sigma_j \tag{0.1}$$

for $v_j \in \mathbb{R}$. Furthermore, we can describe this as the vector space of complex 2×2 matrices with $\operatorname{tr} M_v = 0$ and $M_v^\dagger = M_v$. Both of these properties are preserved by $M_{\boldsymbol{v}} \mapsto gM_{\boldsymbol{v}}g^\dagger$:

$$\begin{aligned} \mathrm{tr} g M_{\boldsymbol{v}} g^\dagger &= \mathrm{tr} g^\dagger g M_{\boldsymbol{v}} = \mathrm{tr} M_{\boldsymbol{v}} = 0 \\ (gM_{\boldsymbol{v}}g^\dagger)^\dagger &= (g^\dagger)^\dagger M_{\boldsymbol{v}}^\dagger g^\dagger = gM_{\boldsymbol{v}}g^\dagger \end{aligned} . \tag{0.2}$$

Finally, the map $F(g)$ acts linearly on $M_{\boldsymbol{v}}$, so that $F : SU(2) \to GL(3, \mathbb{R})$. The only thing left to show to have a representation is that F is a homomorphism. We have

$$F(gh) : M_{\boldsymbol{v}} \mapsto ghM_{\boldsymbol{v}}(gh)^\dagger = ghM_{\boldsymbol{v}}h^\dagger g^\dagger \tag{0.3}$$

which is just the composition of the maps $F(h)$ and $F(g)$ acting on $M_{\boldsymbol{v}}$, so that this is a group homomorphism. More explicitely, if we write the action of $F(g)$ as a matrix acting $\boldsymbol{v}$, the above must be matrix multiplication, i.e. we can then write $F(gh) = F(g)F(h)$.

As defined in the lectures, the adjoint representation of G acts on $\mathfrak{g}$ as

$$\gamma \to g\gamma g^{-1} . \tag{0.4}$$

For $\mathfrak{su}(2)$, $g^{-1} = g^\dagger$ and we can write

$$\gamma = i \sum_j v_j \sigma_j . \tag{0.5}$$

The action of $F(g)$ on the v_j is hence the same as above (despite the extra factor of i) which means that $F(g)$ and the adjoint represention take g to the same elements of $GL(3, \mathbb{R})$, so that we conclude that F is the same as the adjoint representation.

22. Let $\mathbf{q} \in \mathbb{C}^n$ be acted on in the defining (also called 'fundamental') representation of $SU(n)$ and γ in the adjoint representation of $SU(n)$ (this is often expressed as $\mathbf{q}$ 'lives' in the fundamental and γ 'lives' in the adjoint of $SU(n)$.)

    Describe the action of $SU(n)$ on

    i) $\mathbf{v} = \gamma \mathbf{q}$

    ii) $\bar{\mathbf{q}}$

    iii) A matrix Q with components $Q_{ij} = q_i q_j$

    and decide in each case if this defines a representation.

    solution:

    i) $SU(n)$ acts on both γ and $\mathbf{q}$ and sends

    $$\mathbf{v} = \gamma \mathbf{q} \to g\gamma g^{-1} g \mathbf{q} = g\gamma \mathbf{q} = g\boldsymbol{v}\,. \tag{0.6}$$

    The set of all elements $\gamma \mathbf{q}$ forms a vector space which is just $\mathbb{C}^n$, so that this is again the defining representation of $SU(n)$.

    ii) Here we have

    $$\bar{\mathbf{q}} \to \bar{g}\bar{\mathbf{q}} \tag{0.7}$$

    and the set of all $\bar{\mathbf{q}}$ is again $\mathbb{C}^n$. This defines a map from $SU(n)$ to $GL(n, \mathbb{C})$ given by $r(g) = \bar{g}$. Let's check this is a representation by checking it is a homomorphism:

    $$r(gh) = \overline{gh} = \bar{g}\bar{h} = r(g)r(h)\,. \tag{0.8}$$

    iii) In this case the action is (note use of summation convention)

    $$Q_{ij} = q_i q_j \to g_{ik} q_k g_{jl} q_l \tag{0.9}$$

    which in matrix language is

    $$Q \to gQg^T\,. \tag{0.10}$$

To see if this can give us a representation, let us first examine if matrices of the form of Q form a vector space. For this to be the case, we need that there is a $\boldsymbol{q}''$ such that for every $\boldsymbol{q}$ and $\boldsymbol{q}'$ we can write for all i, j

$$q''_i q''_j = q_i q_j + q'_i q'_j \tag{0.11}$$

These are $n(n+1)/2$ independent relations in n complex variables, which have no solution for $n > 1$. (you can also note that setting $i = j$ already fixes all q''_i and we cannot solve the remaining relations.). So matrices of the form Q do not form a vector space and we have nothing to talk about in terms of representations. However, they naturally sit inside the vector space V of all $n \times n$ matrices, or all symmetric $n \times n$ matrices, which are vector spaces. As gh then acts as

$$Q \to ghQ(gh)^T = ghQh^T g^T \tag{0.12}$$

which is just the composition of the maps $r(h)$ and $r(g)$ on Q, so that we have a homomorphism from $SU(n)$ to $GL(V)$.

23. Let $g \in SO(3)$ be given by

$$g = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin(\phi) & \cos(\phi) & 0 \\ 0 & 0 & 1 \end{pmatrix} .$$

Find the action of g in the adjoint representation and describe it using a basis of the vector space $\mathfrak{so}(3)$. As $\mathfrak{so}(3)$ is the same as $\mathbb{R}^3$, we can describe its elements as column vectors after having chosen a basis. Using the basis you have chosen, write the adjoint action as a 3×3 matrix acting on a column vector.

solution:

The Lie algebra $\mathfrak{so}(3)$ of $SO(3)$ is a real three-dimensional vector space with basis (see problem class 2 and 3)

$$\ell_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad \ell_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \ell_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.13}$$

and we can write

$$\gamma = \sum_j a_j \ell_j \tag{0.14}$$

with real a_j for any $\gamma \in \mathfrak{so}(3)$.

For the above element we find the adjoint action

$$\gamma \to g\gamma g^{-1} = g\sum_j a_j \ell_j g^{-1} = \sum_j a_j g\ell_j g^{-1} \,. \tag{0.15}$$

and we need to work out what $g\ell_j g^{-1}$ is to make progress.

Doing a direct computation we find

$$\begin{aligned} g\ell_1 g^{-1} &= \cos\phi\,\ell_1 + \sin\phi\,\ell_2 \\ g\ell_2 g^{-1} &= \cos\phi\,\ell_2 - \sin\phi\,\ell_1 \\ g\ell_3 g^{-1} &= \ell_3 \end{aligned} \tag{0.16}$$

i.e.

$$\sum_j a_j \ell_j \to \sum_j a_j R_{jk} \ell_k \tag{0.17}$$

with

$$R_{jk} = \begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \,. \tag{0.18}$$

Note that the matrix$R$ appearing here is the same as the g we started from! We can hence describe the adjoint action of g on $\boldsymbol{a} = (a_1, a_2, a_3)$ as

$$\boldsymbol{a} \to R^T \boldsymbol{a} \,. \tag{0.19}$$

24. Let G be a Lie group and H be a subgroup of G that is also a Lie group.

   a) Explain why any representation $r(G)$ of G also gives us a representation $r(H)$ of H.

   b) Let's assume $r(G)$ is irreducible. Can you think of an example where the representation $r(H)$ is reducible? Can you think of an example where the representation $r(H)$ is irreducible?

solution:

(a) As we have a representation of G, there is a group homomorphism

$$r : G \to GL(V) \tag{0.20}$$

for some vector space V so that

$$r(gg') = r(g)r(g') \,. \tag{0.21}$$

for all $g, g' \in G$. As H is a subgroup, we can simply restrict ourselves to consider only element $h, h' \in H$. As H is a subgroup, $hh' \in H$ for all $h, h' \in H$. Hence

$$r(hh') = r(h)r(h') \tag{0.22}$$

for all $h, h' \in H$, so that we get a group homomorphism and hence a representation.

(b) To find an example where this becomes reducible, consider the group $SU(3)$ and its subgroup $SU(2)$ of matrices of the form

$$\begin{pmatrix} a & b & 0 \\ -\bar{b} & \bar{a} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.23}$$

with $|a|^2 + |b|^2 = 1$. Taking the defining representation of $SU(3)$, the $SU(2)$ subgroup acts on $\mathbb{C}^3$ as written above, which leaves the subspace of vectors of the form $(0, 0, z_3)$ invariant, so that it is reducible.

To find an example where the representation stays irreducible, consider the defining representation of $O(3)$. This is clearly irreducible. But $O(3)$ has a subgroup $SO(3)$, and restricting the defining representation of $O(3)$ to $SO(3)$ gives the defining representation of $SO(3)$ which is also irreducible.

Here are some things you should discuss with your friends:

1. What is the geometric meaning of the Lie algebra of a Lie group?
2. When and how can you recover a Lie group from its Lie algebra?
3. What is a group representation?