25. Let P be a homogeneous polynomial in two complex variables z_1 and z_2 of degree d, i.e. we can write

$$P(\boldsymbol{z}) = \sum_{k=0}^{d} \alpha_k z_1^k z_2^{d-k} \tag{0.1}$$

for complex numbers α_k.

There is a natural action of $SU(2)$ on $\boldsymbol{z} = (z_1, z_2)$, which is just

$$\boldsymbol{z} \mapsto g\boldsymbol{z} . \tag{0.2}$$

For a polynomial $P(\boldsymbol{z})$, we can then define an action by $SU(2)$ as

$$r_d(g) : P(\boldsymbol{z}) \mapsto P(g^{-1}\boldsymbol{z}) . \tag{0.3}$$

Show that this defines a representation of $SU(2)$.

[remark: in the above formula, g^{-1} does not act on the argument of P but on $\boldsymbol{z}$, i.e. the action on $P(A\boldsymbol{z})$ for a 2×2 matrix A would be $r_d(g) : P(A\boldsymbol{z}) \mapsto P(Ag^{-1}\boldsymbol{z})$.]

26. a) Describe a $U(1)$ subgroup of $SU(2)$. Is $U(1) \times U(1)$ a subgroup of $SU(2)$ as well?

    b) Let A be an element of the vector space that is acted on by the adjoint representation of $SU(2)$. For the $U(1)$ subgroup of $SU(2)$ you identified above, find the action on A and use this to decompose the action of $U(1)$ into irreducible representations.

27. Consider the map $r_\kappa : U(1) \to GL(3, \mathbb{C})$ defined by

$$r_\kappa(e^{i\phi}) = e^{\phi\lambda\kappa}$$

where $\kappa \in \mathbb{C}$ and

$$\lambda = \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

For which values of κ is r_κ a representation of $U(1)$? [hint: think about what happens to eigenvectors of λ]

Here are some things you should discuss with your friends:

1. What is the idea behing reducible and irreducible representations?
2. Under which conditions can you decompose representations into irreducible ones?
3. How are representations of $U(1)$ characterized?