25. Let P be a homogeneous polynomial in two complex variables z_1 and z_2 of degree d, i.e. we can write

$$P(\boldsymbol{z}) = \sum_{k=0}^{d} \alpha_k z_1^k z_2^{d-k} \tag{0.1}$$

for complex numbers α_k.

There is a natural action of $SU(2)$ on $\boldsymbol{z} = (z_1, z_2)$, which is just

$$\boldsymbol{z} \mapsto g\boldsymbol{z}\,. \tag{0.2}$$

For a polynomial $P(\boldsymbol{z})$, we can then define an action by $SU(2)$ as

$$r_d(g) : P(\boldsymbol{z}) \mapsto P(g^{-1}\boldsymbol{z})\,. \tag{0.3}$$

Show that this defines a representation of $SU(2)$.

[remark: in the above formula, g^{-1} does not act on the argument of P but on $\boldsymbol{z}$, i.e. the action on $P(A\boldsymbol{z})$ for a 2×2 matrix A would be $r_d(g) : P(A\boldsymbol{z}) \mapsto P(Ag^{-1}\boldsymbol{z})$.]

solution:

We need to check four things: i) we are acting on a vector space Π_d ii) that this map indeed maps elements of Π_d to Π_d, iii) that it is linear, iv) that it is a group homomorphism from $SU(2)$ to $GL(\Pi_d)$.

i) we can write

$$P(\boldsymbol{z}) = \sum_{k=0}^{d} a_k z_1^k z_2^{d-k}\,.$$

for any such polynomial. Adding two of these or multiplying by a complex number just adds or rescales the a_k, so this defines a vector space which we can call Π_d. You can think of the a_k as the components of the vectors and the monomials as basis vectors. As there are $d+1$ different monomials for a polynomial of degree d, this is a complex vector space of dimension $d+1$.

ii) Here, it is enough to observe that g^{-1} acts linearly on $\boldsymbol{z}$. Hence it preserved the degree of P so that it indeed maps any element of Π_d to another element of Π_d.

iii) Note that

$$r_d(g)(P+Q) = (P+Q)(g^{-1}\boldsymbol{z}) = P(g^{-1}\boldsymbol{z}) + Q(g^{-1}\boldsymbol{z}) = r_d(g)(P) + r_d(g)(Q) \tag{0.4}$$

Hence $r_{d+1}(g)$ acts linearly on Π_d.

iv) Consider the action of $r_d(g)r_d(h)$ on P:

$$r_d(g)r_d(h)P(z) = r_d(g)P(h^{-1}z) = P(h^{-1}g^{-1}\boldsymbol{z}) = P((gh)^{-1}\boldsymbol{z}) = r_d(gh)P\,. \tag{0.5}$$

We hence have a group homomorphism[1]. To see that it is in $GL(\Pi_d)$ you might be concerned about $r_d(g)P = 0$ for some g. That this does not happen follows from the fact that we are talking about a group homomorphism: if $r_d(g)P = 0$ then also $r_d(g^{-1})r_d(g)P = 0$. But $r_d(g^{-1})r_d(g)P = r_d(g^{-1}g)P = P$, which is a contradiction.

As a final comment, note the peculiar $g^{-1}\boldsymbol{z}$ instead of $g\boldsymbol{z}$. The deeper reason for this is that we are acting on the basis vectors (monomials) instead of components as usual.

26. a) Describe a $U(1)$ subgroup of $SU(2)$. Is $U(1)\times U(1)$ a subgroup of $SU(2)$ as well?

b) Let A be an element of the vector space that is acted on by the adjoint representation of $SU(2)$. For the $U(1)$ subgroup of $SU(2)$ you identified above, find the action on A and use this to decompose the action of $U(1)$ into irreducible representations.

solution:

(a) We can simply take matrices of the form

$$g(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix}\,. \tag{0.6}$$

These are in $SU(2)$ for any ϕ and are isomorphic to $U(1)$. There is no subgroup $U(1)\times U(1)$ in $SU(2)$, and we argue as follows. For $U(1)\times U(1)$ there are two generators, one for each $U(1)$ factor, and these generators commute with each other. Otherwise we would be talking about a different group. We are hence looking for two $U(1)$ subgroups of $SU(2)$ which commute. We can write these as

$$e^{i\phi_1\alpha} \text{ and } e^{i\phi_2\beta} \tag{0.7}$$

for some real linear combinations of Pauli matrices $\alpha = \sum_i a_i\sigma_i$ and $\beta = b_i\sigma_i$. Now for these to commute we need

$$\left[e^{i\phi_1\alpha}, e^{i\phi_2\beta}\right] = 0 \tag{0.8}$$

[1] Note that it is for this reason we needed the $g^{-1}\boldsymbol{z}$.

for all values of ϕ_1 and ϕ_2. Taking derivatives with respect to ϕ_1 and ϕ_2 this implies

$$[\alpha, \beta] = 0 \tag{0.9}$$

which is clearly also a sufficient condition. We work out

$$[\alpha, \beta] = [a_i \sigma_i, b_j \sigma_j] = 2i\epsilon_{ijk} a_i b_j \sigma_k \tag{0.10}$$

which only vanishes when $\epsilon_{ijk} a_i b_j = 0$ for all k. The only solution except $\alpha = 0$ or $\beta = 0$ (which is inacceptable since then we don't get $U(1) \times U(1)$) is $a_i = b_i$ for all i. But then we generate the same $U(1)$ twice.

Here is another nice solution that some of you came up with: $U(1) \times U(1)$ has four elements that square to the identity:

$$(1, 1), (-1, 1), (1, -1), (-1, -1)\,.$$

If $U(1) \times U(1)$ were a subgroup of $SU(2)$, at least four such elements must exist in $SU(2)$ as well. By using the general form of $SU(2)$ you can see that $g \in SU(2)$ squaring to the identity implies $|a|^2 = 1$ and $b = 0$, so these matrices must be diagonal, and there only two such matrices which square to the identity.

(b) The vector space we act on in the adjoint is the vector space of matrices A such that $A^\dagger = -A$ and $\mathrm{tr}A = 0$. The action on this is

$$A \rightarrow gAg^\dagger = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} A \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix}\,. \tag{0.11}$$

for the choice of $U(1)$ made above. We find

$$\begin{aligned} A_{11} &\rightarrow A_{11} \\ A_{21} &\rightarrow e^{-2i\phi} A_{21} \\ A_{12} &\rightarrow e^{2i\phi} A_{12} \\ A_{22} &\rightarrow A_{22} \end{aligned} \tag{0.12}$$

Hence $A_{11} = -A_{22}$ is an invariant subspace of charge 0. $A_{12} = -\bar{A}_{21}$ is another invariant subspace of charge 2.

27. Consider the map $r_\kappa : U(1) \rightarrow GL(3, \mathbb{C})$ defined by

$$r_\kappa(e^{i\phi}) = e^{\phi\lambda\kappa}$$

where $\kappa \in \mathbb{C}$ and

$$\lambda = \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

For which values of κ is r_κ a representation of $U(1)$? [hint: think about what happens to eigenvectors of λ]

solution: First note that

$$r(e^{i\phi})r(e^{i\psi}) = e^{\phi\lambda\kappa}e^{\psi\lambda\kappa} = e^{(\phi+\psi)\lambda\kappa}$$

so this looks like a homomorphism. Parametrizing $U(1)$ as we did, we also need to make sure that $r(e^{2\pi i}) = \mathbb{1}$. This is not obvious immediately.

We know that if this is a representation, we can decompose it into irreducible representations, which are one-dimensional. We are hence looking for three invariant subspace of $\mathbb{C}^3$, which we can construct from eigenvectors of λ. Whenever

$$\lambda \boldsymbol{v} = c\boldsymbol{v}$$

we find that for any vector $a\boldsymbol{v}$ proportional to $\boldsymbol{v}$:

$$e^{\phi\lambda\kappa}a\boldsymbol{v} = e^{\phi c\kappa}a\boldsymbol{v}\,. \tag{0.13}$$

The eigenvalues of λ are $\pm i\sqrt{2}$ and 0, and the eigenvectors are $(1, \pm\sqrt{2}, 1)$ and $(-1, 0, 1)$. The subspace spanned by $(-1, 0, 1)$ carries a trivial representation, but the other two subspaces are acted on by

$$e^{\phi i\sqrt{2}\kappa}\,. \tag{0.14}$$

We hence need $\sqrt{2}\kappa = n$ with $n \in \mathbb{Z}$ for this to be a representation, which implies

$$\kappa = n/\sqrt{2}\,.$$

Here are some things you should discuss with your friends:

1. What is the idea behing reducible and irreducible representations?
2. Under which conditions can you decompose representations into irreducible ones?
3. How are representations of $U(1)$ characterized?