28. Consider the Lie group G of upper triangular 2×2 matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} |a, b, c \in \mathbb{R}, ac \neq 0 \right\}$$

   a) Let $\boldsymbol{v} \in \mathbb{R}^3$, $\boldsymbol{v} = (v_1, v_2, v_3)$. Define an action of G on $\boldsymbol{v}$ by writing

$$v_m := \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix}$$

   and letting $g \in G$ act as

$$r(g)v_m := g v_m g^{-1} \, .$$

   Convince yourself that this is a representation of G. Write the action of g in $\boldsymbol{v}$ defined above in terms of a 3×3 matrix acting on $\boldsymbol{v}$:

$$r(g)\boldsymbol{v} = M(g)\boldsymbol{v}$$

   for a 3×3 matrix $M(g)$ acting on the vector $\boldsymbol{v} \in \mathbb{R}^3$ in the usual way.

   b) Writing elements of the representation $r(G)$ in terms of the matrices $M(g)$, work out the associated representation ρ of the Lie algebra $\mathfrak{g}$ of G.

   c) Check that they give a Lie algebra representation of the Lie algebra $\mathfrak{g}$ of G (see problem 20), i.e. find a Lie algebra homomorphism between the Lie algebra $\mathfrak{g}$ of G and the Lie algebra representation $\rho(\mathfrak{g})$ associated with $r(G)$.

29. Let $\{t_a\}$ be the basis of a Lie algebra $\mathfrak{g}$ and $f_{ab}{}^c$ the structure constants which obey $[t_a, t_b] = f_{ab}{}^c t_c$. Define

$$(\rho_{adj}(t_a))^b{}_c = f_{ac}{}^b \, .$$

   a) Check that the above defines a representation of γ.

   b) Show that the adjoint action in the basis $\{t_a\}$ is given by the matrices $\rho_{adj}(t_a)$ with components $f_{ac}{}^b$ by showing that

$$ad(t_a)(\gamma^b t_b) = (\rho_{adj}(t_a))^b{}_c \gamma^c t_b \, .$$

Here are some things you should discuss with your friends:

1. What is Lie algebra representation?
2. How does a group representation give rise to a Lie algebra representation?
3. What can we say about complex representations of $SU(2)$?