28. Consider the Lie group G of upper triangular 2×2 matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{R}, ac \neq 0 \right\} \tag{0.1}$$

a) Let $\boldsymbol{v} \in \mathbb{R}^3$, $\boldsymbol{v} = (v_1, v_2, v_3)$. Define an action of G on $\boldsymbol{v}$ by writing

$$v_m := \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} \tag{0.2}$$

and letting $g \in G$ act as

$$r(g)v_m := g v_m g^{-1} \,. \tag{0.3}$$

Convince yourself that this is a representation of G. Write the action of g on $\boldsymbol{v}$ defined above in terms of a 3×3 matrix acting on $\boldsymbol{v}$:

$$r(g)\boldsymbol{v} = M(g)\boldsymbol{v} \tag{0.4}$$

for a 3×3 matrix $M(g)$ acting on the vector $\boldsymbol{v} \in \mathbb{R}^3$ in the usual way.

b) Writing elements of the representation $r(G)$ in terms of the matrices $M(g)$, work out the associated representation ρ of the Lie algebra $\mathfrak{g}$ of G.

c) Check that they give a Lie algebra representation of the Lie algebra $\mathfrak{g}$ of G (see problem 20), i.e. find a Lie algebra homomorphism between the Lie algebra $\mathfrak{g}$ of G and the Lie algebra representation $\rho(\mathfrak{g})$ associated with $r(G)$.

solution:

(a) First of all, this is a map that maps a matrix of the form v_m to another one of this form as product of upper triangular 2×2 matrices are again upper triangular as seen in problem 18. Furthermore it acts on the v_i linearly:

$$g(v_m + v'_m)g^{-1} = g(v_m)g^{-1} + g(v'_m)g^{-1} \tag{0.5}$$

so r defines a map from G to $GL(3, \mathbb{R})$. What is left to check is that r is a group homomorphism. We work out

$$r(gh)v_m = ghv_m(gh)^{-1} = ghv_m h^{-1} g^{-1} = r(g)r(h)v_m \,. \tag{0.6}$$

which shows it is. Hence this is a real three-dimensional representation of G. It is not injective as both g and $-g$ are mapped to $\mathbb{1} \in GL(3, \mathbb{R})$.

Note that this is simply the adjoint representation.

Let us work out explicitely how $r(g)$ acts on v_m:

$$v_m = \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} \to v'_m = g v_m g^{-1} = \frac{1}{ac}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix}$$
$$= \frac{1}{ac}\begin{pmatrix} acv_1 & -v_1 ab + a^2 v_2 + abv_3 \\ 0 & acv_3 \end{pmatrix} \tag{0.7}$$

Hence

$$\begin{aligned} v_1 &\to v_1 \\ v_2 &\to -b/cv_1 + a/cv_2 + b/cv_3 \\ v_3 &\to v_3 \end{aligned} \tag{0.8}$$

We can write this as the action of a 3×3 matric on a column vector as

$$\boldsymbol{v} \to \boldsymbol{v}' = M(g)\boldsymbol{v} = \begin{pmatrix} 1 & 0 & 0 \\ -b/c & a/c & b/c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{0.9}$$

(b) Using the family of paths $a = e^{xt}$, $b = ty$, $c = e^{zt}$, $x, y, z \in \mathbb{R}$, we get

$$M\left(g(t)\right) = \begin{pmatrix} 1 & 0 & 0 \\ -tye^{-zt} & e^{t(x-z)} & tye^{-tz} \\ 0 & 0 & 1 \end{pmatrix} \tag{0.10}$$

and

$$\frac{\partial}{\partial t} M\left(g(t)\right)\bigg|_{t=0} = \begin{pmatrix} 0 & 0 & 0 \\ -y & x-z & y \\ 0 & 0 & 0 \end{pmatrix} \tag{0.11}$$

This defines the associated Lie algebra representation ρ. Hence for every γ in $\mathfrak{g}$ we can write

$$\rho(\gamma) = x\rho(\ell_x) + y\rho(\ell_y) + z\rho(\ell_z) \tag{0.12}$$

where

$$\rho(\ell_x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho(\ell_y) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho(\ell_z) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.13}$$

However, these matrices are not all linearly independent, i.e. the vector space they span is just two-dimensional as $\rho(\ell_z) = -\rho(\ell_x)$.

We can present a general element as

$$\rho(\gamma) = u\rho(\ell_x) + y\rho(\ell_y) \tag{0.14}$$

Note that using different paths here might give you a different basis of the Lie algebra.

The matrices above satisfy

$$[\rho(\ell_x), \rho(\ell_y)] = \rho(\ell_y) \quad [\rho(\ell)_x, \rho(\ell)_z] = 0 \quad [\rho(\ell)_z, \rho(\ell)_y] = -\ell_y\,. \tag{0.15}$$

(c) Using the paths above in problem 18 you find that a general element of the Lie algebra of G can be written as

$$x\begin{pmatrix}1 & 0\\ 0 & 0\end{pmatrix} + y\begin{pmatrix}0 & 0\\ 0 & 1\end{pmatrix} + z\begin{pmatrix}0 & 1\\ 0 & 0\end{pmatrix} \equiv x\ell_x + y\ell_y + z\ell_z \tag{0.16}$$

for $x, y, z \in \mathbb{R}$. The algebra of the ℓ is

$$[\ell_x, \ell_y] = \ell_y \quad [\ell_x, \ell_z] = 0 \quad [\ell_z, \ell_y] = -\ell_y\,. \tag{0.17}$$

The same relations are obeyed by $\rho(\ell_x)$, $\rho(\ell_y)$ and $\rho(\ell_z)$ as it must be.

29. Let $\{t_a\}$ be the basis of a Lie algebra $\mathfrak{g}$ and $f_{ab}{}^c$ the structure constants which obey $[t_a, t_b] = f_{ab}{}^c t_c$. Define

$$\left(\rho_{adj}(t_a)\right)^b{}_c = f_{ac}{}^b\,. \tag{0.18}$$

a) Check that the above defines a representation of γ by showing that $[\rho_{adj}(t_a), \rho_{adj}(t_b)] = \rho_{adj}([t_a, t_b])$. [hint: use the Jacobi identity written in terms of structure constants.]

b) Show that the adjoint action in the basis $\{t_a\}$ is given by the matrices $\rho_{adj}(t_a)$ with components $f_{ac}{}^b$ by showing that

$$ad(t_a)(\gamma^b t_b) = \left(\rho_{adj}(t_a)\right)^b{}_c \gamma^c t_b\ . \tag{0.19}$$

solution:

a) We need to show that, abbreviating $\rho_{adj} = \rho$,

$$[\rho(t_a), \rho(t_b)] = \rho([t_a, t_b])\,, \tag{0.20}$$

so let's express both sides in terms of $f_{ab}{}^c$ using the definition of ρ_{adj}:

$$\rho([t_a, t_b])^e{}_c = \rho(f_{ab}{}^d t_d)^e{}_c = f_{ab}{}^d f_{dc}{}^e \tag{0.21}$$

and

$$([\rho(t_a), \rho(t_b)])^e{}_c = \rho(t_a)^e{}_d\rho(t_b)^d{}_c - \rho(t_b)^e{}_d\rho(t_a)^d{}_c = f_{ad}{}^e f_{bc}{}^d - f_{bd}{}^e f_{ac}{}^d \tag{0.22}$$

so that after using $f_{ab}{}^c = -f_{ba}{}^c$

$$[\rho(t_a), \rho(t_b)] - \rho([t_a, t_b]) = -f_{ca}{}^d f_{db}{}^e - f_{bc}{}^d f_{da}{}^e - f_{ab}{}^d f_{dc}{}^e = 0 \tag{0.23}$$

using the Jacobi identity.

b) We work out

$$ad(t_a)(\gamma^b t_b) = [t_a, \gamma^b t_b] = \gamma^b f_{ab}{}^c t_c = \rho_{adj}(t_a)^c_b \gamma^b t_c\,, \tag{0.24}$$

i.e. the components γ^b of any element of $\mathfrak{g}$ are acted on as

$$\gamma^b \to \rho_{adj}(t_a)^b{}_c \gamma^c\,. \tag{0.25}$$

Here are some things you should discuss with your friends:

1. What is Lie algebra representation?
2. How does a group representation give rise to a Lie algebra representation?
3. What can we say about complex representations of $SU(2)$?