1. Lets first discuss the **key** concepts of the EP lectures and how to check/show various basic things. If you were to make an exam, you might come up with 'bread and butter' questions that require you to use what is written below. Alternatively, a good exercise is to design a cheat sheet which contains the crucial things to have taken away from these lectures. The answers to the questions below are things you might want to include in such a cheat sheet.

   (a) What is the Lorentz group?
   answer: Linear maps on $\mathbb{R}^4$ that leave $-(x^0)^2 + (x^1)^2 + (x^2)^2 + (x^3)^2$ invariant.

   (b) What's the deal with upper/lower indices? **answer:** Define $x_i = x^i$ and $x_0 = -x^0$ (or write $x_\mu = \eta_{\mu\nu} x^\nu$) so that
   $$x_\mu x^\mu \tag{1}$$
   is the invariant.

   (c) What is a spinor?
   answer: This is an object that lives in a vector space acted on by a representation of the double cover $SL(2,\mathbb{C})$ of the proper orthochronous Lorentz group $L_+^\uparrow$.
   $$\Psi \to \Lambda_{1/2}\Psi = \exp(\theta_{\mu\nu} S^{\mu\nu})\Psi \tag{2}$$
   where $S^{\mu\nu} = \frac{1}{4}[\gamma^\mu, \gamma^\nu]$. Recall that the $S^{\mu\nu}$ are a Lie algebra representation of the Lie algebra of the Lorentz group, but their exponentiation is not a group representation of $L_+^\uparrow$. This is the same situation as the relationship between $SO(3)$ and the **2** representation of $SU(2)$.

   (d) How do you find the equations of motion from a field theory action?
   answer:
   For $S = \int d^4x \mathcal{L}(\phi, \partial_\mu \mathcal{L})$, work out the Euler-Lagrange eqs:
   $$0 = \partial_\mu \frac{\partial}{\partial \partial_\mu \phi_I}\mathcal{L} - \frac{\partial}{\partial \phi_I}\mathcal{L} \tag{3}$$
   for every field ϕ_I. Note that the index I is supposed to run over all real fields contained in the action. For complex fields, we can treat either the real and imaginary parts as independent fields, or do the same with ϕ and $\bar{\phi}$.

   (e) What is Noether's theorem?
   answer:

For a symmetry of S under action of a Lie group, there is a conserved current

$$j^\mu := [\rho(\gamma)\phi]_I \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_I)} \tag{4}$$

Here $\rho(\gamma)\phi$ is the Lie algebra representation associated to the group action on $\phi \in V$. Note there is a sum over I in the above expression which runs over all fields (which transform non-trivially). For a complex field, you get 2 contributions in the sum, e.g. one from ϕ and one from $\bar{\phi}$.

(f) How do you check if an action is Lorentz invariant?

answer:

As explained in the lectures, what we mean by a field theory to be Lorentz invariant is that any solution $\phi(\boldsymbol{x})$ implies solutions $\phi(\Lambda^{-1}\boldsymbol{x})$. For an action, this means that replacing the arguments of the fields, but not derivatives or the integration variables. However, vector/spinor/etc fields also have an 'exterior' transformation, e.g.

$$A^\mu(\boldsymbol{x}) \to \Lambda^\mu{}_\nu A^\nu(\Lambda^{-1}\boldsymbol{x}) \tag{5}$$

Now when replacing coords in the arguments of the fields and using the chain rule, derivatives **do** effectively transform as indicated by their indices, e.g.

$$\partial_\mu \phi(\Lambda^{-1}\boldsymbol{x}) = \left(\Lambda^{-1}\right)^\nu{}_\mu \frac{\partial}{\partial y^\nu}\phi(\boldsymbol{y})\,. \tag{6}$$

This means that at the end of the day

$$S = \int d^4x \partial_\mu \phi \partial^\mu \phi \tag{7}$$

is Lorentz invariant for the 'naive' reason that we have contracted all indices.

(g) Why are Maxwell's equation's Lorentz invariant and what is the action they follow from ?

answer:

The action is

$$S = \int d^4x - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} + A_\mu J^\mu \tag{8}$$

and $F_{\mu\nu}$ contains electric and magnetic fields. The field strength needs to be understood as a function of A_μ

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{9}$$

and varying the action produces

$$\partial_\nu F^{\mu\nu} = J^\mu \,. \tag{10}$$

These are the inhom. Maxwell eqs. The hom Maxwell eqs are the identity $\epsilon^{\mu\nu\rho\sigma}\partial_\nu F_{\rho\sigma} = 0$.

(h) What is abelian gauge invariance and how do you check something is gauge invariant?

answer:

The transformation $A_\mu \to A_\mu + \partial_\mu \alpha(\boldsymbol{x})$ does not change $F_{\mu\nu}$. There can in general be other fields participating in gauge transformations, e.g. $\phi \to e^{i\alpha}\phi$. To check gauge invariance just apply the gauge transformation to all the fields.
Be careful with derivatives as α depends on $\boldsymbol{x}$.

(i) What is a covariant derivative?

For $\phi \to e^{i\alpha}\phi$ the covariant derivative is supposed to transform the same way

$$D_\mu \phi \to e^{i\alpha} D_\mu \phi \tag{11}$$

This is achieved by letting

$$D_\mu = \partial_\mu \phi - i A_\mu \phi \tag{12}$$

(j) What is a non-abelian gauge theory? How is a gauge invariant action constructed here?

For a field in the defining representation

$$\phi \to e^{i\alpha}\phi = g\phi \tag{13}$$

with $\alpha \in \mathfrak{g}$ for a Lie algebra $\mathfrak{g}$ with group G and

$$A_\mu \to g(A_\mu + i\partial_\mu)g^{-1} \tag{14}$$

for the gauge field.

$$F_{\mu\nu} = i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{15}$$

transforms as

$$F_{\mu\nu} \to e^{i\alpha} F_{\mu\nu} e^{-i\alpha} \tag{16}$$

and an invariant action is

$$S = \int d^4x \;\; -\frac{1}{2g_{YM}^2}\mathrm{tr}\left(F_{\mu\nu}F^{\mu\nu}\right) \tag{17}$$

for a constant g_{YM}. For a field in representation r we have

$$D_\mu \phi = \partial_\mu \phi - i A_\mu \phi := \left(\mathbb{1}_r \partial_\mu - i A^a_\mu t^{(r)}_a\right)\phi \,, \tag{18}$$

2. A non-abelian gauge field $A_\mu = A^a_\mu t_a$ has field strength

$$F_{\mu\nu} = F^a_{\mu\nu} t_a := \partial_\mu A_\nu - \partial_\nu A_\mu - i g_{YM} [A_\mu, A_\nu] \; .$$

(Notice the unusual normalization, with the Yang-Mills coupling constant g appearing inside the field strength.)

(a) Write the components $F^a_{\mu\nu}$ of the field strength in terms of the components A^a_μ of the gauge field and the structure constants $f_{ab}{}^c$ of the Lie algebra of the gauge group, which has Lie bracket

$$[t_a, t_b] = i f_{ab}{}^c t_c \; .$$

(b) Working in a normalization where $\mathrm{tr}(t_a t_b) = \frac{1}{2}\delta_{ab}$, write the Lagrangian density

$$\mathcal{L} = -\frac{1}{2} \mathrm{tr}(F_{\mu\nu} F^{\mu\nu})$$

as a polynomial in g_{YM},

$$\mathcal{L} = \mathcal{L}_0 + g_{YM} \mathcal{L}_1 + g_{YM}^2 \mathcal{L}_2 \; ,$$

and find explicit expressions for $\mathcal{L}_0$, $\mathcal{L}_1$, $\mathcal{L}_2$ in terms of the components A^a_μ of the gauge field and the structure constants $f_{ab}{}^c$ of the Lie algebra.

solution:

(a) We work out

$$\begin{aligned} F^a_{\mu\nu} t_a = F_{\mu\nu} &= \partial_\mu A^a_\nu t_a - \partial_\nu A^a_\mu t_a - i g_{YM} [A^b_\mu t_b, A^c_\nu t_c] \\ &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) t_a - i g_{YM} A^b_\mu A^c_\nu i f_{bc}{}^a t_a \\ &= \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_{YM} A^b_\mu A^c_\nu f_{bc}{}^a\right) t_a \end{aligned} \tag{19}$$

(b) What we need to do is expand the action by writing $F_{\mu\nu}$ in terms of A^a_μ. This can be evaluated in a straightforward way using the result from the first part of this exercise:

$$\begin{aligned} \mathcal{L} &= -\frac{1}{2} \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) = -\frac{1}{2} F^a_{\mu\nu} F^{b\mu\nu} \mathrm{tr}(t_a t_b) = -\frac{1}{4} F^a_{\mu\nu} F_a{}^{\mu\nu} \\ &= -\frac{1}{4} \left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu\right) \left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) \\ &+ g_{YM} (-\tfrac{1}{2} f_{bc}{}^a) A^b_\mu A^c_\nu \left(\partial^\mu A^\nu_a - \partial^\nu A^\mu_a\right) \\ &+ g_{YM}^2 (-\tfrac{1}{4}) f_{bc}{}^a f^{de}{}_a A^b_\mu A^c_\nu A^\mu_d A^\nu_e \end{aligned} \tag{20}$$

here we have taken the convention of lowering/raising Lie algebra indices with δ_{ab} (so you might as well put them up/down whereever you want, I have only written it like this to make it easier to read). You can now read off $\mathcal{L}_0, \mathcal{L}_1, \mathcal{L}_2$ from the above expression. It is not required to simplify $\mathcal{L}_1$ as

$$\mathcal{L}_1 = -f_{bc}{}^a A^b_\mu A^c_\nu \partial^\mu A^\nu_a \tag{21}$$

but you could if you liked.

3. Let Ψ be a Dirac spinor and γ^μ the Dirac matrices.

   (a) Let $S := \gamma^\mu s_\mu$ for a Lorentz vector s_μ. Show that S^2 is proportional to the identity matrix and find the constant of proportionality.

   (b) How does

   $$\Gamma^{\mu\nu} := \bar{\Psi}\gamma^\mu\gamma^\nu\Psi$$

   transformation under elements of the Lorentz group?

   (c) Write Ψ in terms of Weyl spinors by setting $\Psi = (\Psi_L, \Psi_R)$ and find the behaviour of Weyl spinors under Lorentz transformations.

   (d) For Ψ_L and χ_L two left-handed Weyl spinors with components $(\Psi_L)_I$ and $(\chi_L)_J$ find the behavior under rotations for

   $$M_{IJ} := (\Psi_L)_I(\bar{\chi}_L)_J\,.$$

   Decompose the associated representation in terms of irreducible representations.

solution:

(a) We need to use the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$:

$$S^2 = \gamma^\mu s_\mu \gamma^\nu s_\nu = \gamma^\mu\gamma^\nu s_\nu s_\mu = \frac{1}{2}\{\gamma^\mu, \gamma^\nu\} s_\nu s_\mu = s_\nu s_\mu \eta^{\mu\nu} = s^\mu s_\mu \tag{22}$$

(b) Here we need to recall that Ψ transforms with $\Lambda_{1/2}$ and that

$$\Lambda^\dagger_{1/2}\gamma^0 = \gamma_0 \Lambda^{-1}_{1/2} \tag{23}$$

as well as

$$\Lambda^{-1}_{1/2}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu \tag{24}$$

where $\Lambda^\mu{}_\nu$ is the Lorentz transformation associated with $\Lambda_{1/2}$. Putting it all together we find

$$\begin{aligned}\Gamma^{\mu\nu} \to \Psi^* \Lambda_{1/2}^\dagger \gamma^0 \gamma^\mu \gamma^\nu \Lambda_{1/2} \Psi &= \Psi^* \gamma^0 \Lambda_{1/2}^{-1} \gamma^\mu \gamma^\nu \Lambda_{1/2} \Psi \\ &= \Psi^* \gamma^0 \Lambda_{1/2}^{-1} \gamma^\mu \Lambda_{1/2} \Lambda_{1/2}^{-1} \gamma^\nu \Lambda_{1/2} \Psi \\ &= \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \bar{\Psi} \gamma^{\mu'} \gamma^{\nu'} \Psi = \Lambda^\mu{}_{\mu'} \Lambda^\nu{}_{\nu'} \Gamma^{\mu'\nu'}\end{aligned} \tag{25}$$

This hence transforms as the indices would suggest.

(c) We need to recall that

$$\Psi \to e^{S^{\mu\nu}\theta_{\mu\nu}} \Psi \tag{26}$$

and that $S^{\mu\nu}$ is block diagonal:

$$S^{0i} = \frac{1}{2} \begin{pmatrix} \sigma_i & 0 \\ 0 & -\sigma_i \end{pmatrix} \tag{27}$$

and

$$S^{ij} = \frac{i}{2} \epsilon_{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix} \tag{28}$$

Hence

$$\Psi_{L/R} \to \exp\left(\pm \frac{1}{2} \sigma_i \theta_{0i} + \frac{i}{2} \epsilon_{ijk} \sigma_k \theta_{ij} \right) \Psi_{L/R} \tag{29}$$

(there are sums over i, j, k here).

(d) Using the result from the last part and recalling that only $\theta_{ij} \neq 0$ for rotations we get

$$(\Psi_L)_I (\bar{\chi}_L)_J \to \left[\exp\left(i\alpha_i \sigma_i\right)\right]_{IK} \left[\exp\left(-i\alpha_i \bar{\sigma}_i\right)\right]_{JM} (\Psi_L)_K (\bar{\chi}_L)_M \tag{30}$$

where we have rewritten $\frac{i}{2}\epsilon_{ijk}\sigma_l\theta_{ij} = i\alpha_i\sigma_i$. I.e. they transform in the 2 and $\bar{2}$ of $SU(2)$ under rotations. Hence letting $M_{IJ} = (\Psi_L)_I(\bar{\chi}_L)_J$ we simply get

$$M \to g M g^\dagger \tag{31}$$

for $g \in SU(2)$.

Now one needs to realize that to get a representation we need a vector space, i.e. we need to allow arbitrary linear combinations of different M, i.e. this is not the same as the adjoint but a tensor product $\bar{2} \otimes 2$. This problem was treated in the lectures where we showed that this is decomposed as $\bar{2} \otimes 2 = 1 \oplus 3$. Alternatively you can reconstruct this by observing that the trace of M is a singlet, while the rest gives us a complex traceless matrix, which is the same as the complexifiction of the Lie algebra of $SU(2)$, i.e. a complexification of the adjoint.