1. Lets first discuss the **key** concepts of the MM lectures and how to check/show various basic things. If you were to make an exam, you might come up with 'bread and butter' questions that require you to use what is written below. Alternatively, a good exercise is to design a cheat sheet which contains the crucial things to have taken away from these lectures. The answers to the questions below are things you might want to include in such a cheat sheet.

   (a) How can you show something is a Lie group?

   answer:

   - Check it is a group.
   - It is a differentiable manifold. We only covered matrix Lie group, key theorem: (topologically) closed subgroups of $GL(n,\mathbb{R})$ (or $GL(n,\mathbb{C})$)are Lie groups.

   (b) How do I check something is a subgroup?

   answer: For a group G, say we take some subset of elements $H = \{h\}$. If using the group composition of G, H is again a group, H is called a subgroup of G. To check this, you need to make sure that

   - id$\in H$
   - $h^{-1} \in H$ if $h \in H$
   - for any h_1 and h_2 in H, it follows that $h_1 h_2 \in H$

   (c) How to find the Lie algebra $\mathfrak{g}$ of a Lie group G?

   answer:

   The Lie algebra is the tangent space at the identity with the commutator as the algebra composition.

   - For every path $g(t)$ in G, compute
   $$\frac{\partial}{\partial t} g(t)|_{t=0} \quad (1)$$
   where $g(0) = \text{id}$.
   - The dimension of the Lie algebra (as a vector space) is equal to the dimension of the group (as a manifold).
   - compute $[\alpha, \beta]$ for all $\alpha, \beta \in \mathfrak{g}$

   (d) How do you show r is a representation?

   answer: For a vector space V, a representation is a group homomorphism $r : G \to GL(V)$. Check:

   - $r(g)$ acts as a linear invertible map on V (i.e. r maps to $GL(V)$).
   - $r(gh) = r(g)r(h)$

(e) How do you get the Lie algebra representation ρ corresponding to a group representation?

answer:

- Compute
$$\frac{\partial}{\partial t} r\left(g(t)\right)|_{t=0} \tag{2}$$
where $g(0) = \text{id}$.
- It follows that
$$[\rho(\alpha), \rho(\beta)] = \rho([\alpha, \beta]) \tag{3}$$
(defining property of Lie algebra representation)

(f) How to show a representation is irreducible?

answer:

check there are no invariant subspaces other than V and $\emptyset$.

(g) What are the complex irreducible representations of $U(1)$ and $SU(2)$? What irreducible representations of other Lie groups do you know?

answer:

- $U(1)$: $r(e^{i\phi}) = e^{qi\phi} = g^q$, $q \in \mathbb{Z}$.
- $SU(2)$: there is a rep on $\mathbb{C}^{n+1}$ for every $n \in \mathbb{Z}$. Act with $g \in SU(2)$ on z_1, z_2 and define r_{n+1} by the action of g^{-1} on
$$P(z_1, z_2) = \sum_{k=0}^{n} z_1^k z_2^{n-k} a_k \tag{4}$$
- For every group we discussed, there is the defining representation and the adjoint representation acting on the Lie algebra of the group.

2. Let $\mathbb{C}^*$ be the set of non-zero complex numbers, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$.

(a) Explain why $\mathbb{C}^*$ is a Lie group if we use multiplication as the group composition. Is $\mathbb{C}^*$ also a Lie group if we use addition instead ?

answer:

Clearly this is a group under multiplication as introduced in the lecures. It is a Lie group because we can cover all of $\mathbb{C}^*$ by a single coordinate chart in which we map to $\mathbb{R}^2$, giving it the structure of a differentiable manifold. The group composition (complex multiplication) is a differentiable map. Note that $\mathbb{C}^* = GL(1, \mathbb{C})$, so this being a Lie group is also already covered by general theorems, and quoting them here would be enough.

The second part is a bit of a trick question. Under addition this is not even a group as we lack the identity element.

(b) Find the Lie algebra $\mathfrak{c}^*$ of $\mathbb{C}^*$. Is the exponential map surjective ?

answer:

The first part can be found in the lecture notes. Writing paths $g = e^{tx}$ shows that $\mathfrak{c}^* = \mathbb{C}$. Furthermore $[x, y] = 0$ for all $x, y \in \mathbb{C}$. We can write any element g of $\mathbb{C}^*$ as $g = e^x$ for $x \in \mathbb{C}$, so the exponential map is surjective.

(c) For $g \in \mathbb{C}^*$ define a map

$$r_k : g \mapsto g^k$$

for $k \in \mathbb{C}$. Find the values of k for which this is a representation of $\mathbb{C}^*$ on $\mathbb{C}$.

answer:

We need this to be a group homomorphism from $\mathbb{C}^*$ to $GL(1, \mathbb{C}) = \mathbb{C}^*$. As $e^x = 1$ for $x = 2\pi i n$ and $n \in \mathbb{Z}$, it follows that

$$g^k = e^{kx} = 1 \tag{5}$$

for $x = 2\pi i$, so that $k \in \mathbb{Z}$. This can also be seen using the homomorphism property, observe that

$$e^x e^y = 1 \tag{6}$$

if $x + y = 2\pi i$, so that

$$r_k(e^x e^y) = r_k(1) = 1 \tag{7}$$

as well. Hence

$$e^{kx} e^{ky} = e^{k(x+y)} = 1 \tag{8}$$

and $k(x + y) = 2\pi i$ and we need $k \in \mathbb{Z}$.

(d) For $g = \exp(\gamma)$ and $\gamma \in \mathfrak{c}^*$ define $\rho_k(\gamma)$ by

$$r_k(g) = \exp\left(\rho_k(\gamma)\right)$$

Find the map $\rho_k(\gamma)$. For which values of k is this a representation of the Lie algebra $\mathfrak{c}^*$?

answer:

This is rather simple as $\rho_k(x) = kx$ follows from the definition. Rescaling a complex number is a linear map which preserves $[x, y] = 0$, so this gives us an algebra representation for every $k \in \mathbb{C}$.

3. Let G be the Lie group $SU(3)$.

   (a) For $g \in G$ and ϕ in the adjoint representation of G, show that for an arbitrary polynomial $Q(\phi)$ with complex coefficients, $\operatorname{tr} Q(\phi)$ is invariant under the group action.

   answer:

   We observe that

   $$\mathrm{tr}\phi^k \to \mathrm{tr} g\phi g^\dagger g\phi g^\dagger \cdots g\phi g^\dagger = \mathrm{tr} g\phi^k g^\dagger = \mathrm{tr}\phi^k \tag{9}$$

   is invariant. For a polynomial Q we then have

   $$Q(\phi) = \sum_k c_k \mathrm{tr}\phi^k \to \sum_k c_k \mathrm{tr}\phi^k = Q(\phi) \tag{10}$$

   (b) Consider the vector space V_d of complex homogeneous polynomials of degree d in 3 variables $\vec{q} = (q_1, q_2, q_3)$:

   $$P(\vec{q}) = \sum_{i=0}^{d} \sum_{j=0}^{d-i} \alpha_{ij} q_1^i q_2^j q_3^{d-i-j}$$

   (here α_{ij} are complex numbers). Show that acting on $\vec{q}$ as $\vec{q} \to g^{-1}\vec{q}$ defines a representation of $SU(3)$ on V_d. In the following, we denote this representation by $F_d(h)$.

   answer:

   Such polynomials span a vector space V_d and the action is a linear map on the coefficients, it is hence in $GL(V_d)$. We need to see it is a homomorphism. Let us write

   $$F_d(g) : P(\vec{q}) \to P(g^{-1}\vec{q})\,.$$

   Clearly $F_d(\mathbb{1}) = \mathbb{1}$. Furthermore

   $$F_d(gh)P = P\left((gh)^{-1}\vec{q}\right) = P\left(h^{-1}g^{-1}\vec{q}\right) = F_d(g)P(h^{-1}\vec{q}) = F_d(g) \circ F_d(h)P(\vec{q}) \tag{11}$$

   so we have a homomorphism.

(c) Show that $SU(2)$ is a subgroup of $SU(3)$ and that $F_d(g)$ defines a representation of the $SU(2)$ subgroup you have identified as well.

answer:

First we need to identify $SU(2)$. We can simply use block-diagonal matrices

$$g_{SU(3)} = \begin{pmatrix} a & b & 0 \\ c & d & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} g_{SU(2)} & 0 \\ 0 & 1 \end{pmatrix} \tag{12}$$

where

$$g_{SU(2)} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{13}$$

is in $SU(2)$. Clearly $g_{SU(2)} \in SU(2)$ implies that $g_{SU(3)} \in SU(3)$.

The embedding above implies that we can act with $SU(2)$ on V_d as well using the same action and the argument above shows that this is a representation as well.

(d) Show that $F_d(h)$ is an irreducible representation or decompse it into irreducible representations.

answer:

The crucial observation is that the action of $SU(2)$ leaves subspaces $V_{d,k}$ invariant where $V_{d,k}$ are spaces of polynomials of the form

$$\sum_{i=0}^{d-k} \alpha_i q_1^i q_2^{d-i-k} q_3^k \tag{14}$$

as the action on q_3 is trivial. These are hence isomorphic to homogeneous polynomials in two variables and of degree $d-k$. The dimension of $V_{d,k}$ is $d-k+1$ and we have shown in the lecture that the action on $V_{d,k}$ gives irreducible representations r_{d-k} of $SU(2)$. The decomposition we are after is hence

$$F_d = \oplus_{k=0}^{d} r_{d-k} \tag{15}$$