Problem class 1

1) (a) Show that $SO(3)$ is a group using matrix multipication as the group composition.

(b) Verify that acting with $g \in SO(3)$ on a vector $\boldsymbol{v} \in \mathbb{R}^3$ as $\boldsymbol{v} \to g\boldsymbol{v}$ implies that the lengh of $\boldsymbol{v}$ stays invariant.

(c) For a matrix $g = e^\gamma$, what conditions do we need to put on γ such that $g \in SO(3)$?

(d) The group $O(3)$ is the group of matrices g which map a vector $\boldsymbol{v} \in \mathbb{R}^3$ to

$$\boldsymbol{v} \mapsto S\boldsymbol{v} \tag{0.0.1}$$

such that the inner form on $\mathbb{R}^3$, $\boldsymbol{v} \cdot \boldsymbol{v} = \sum_i v_i^2$, stays invariant.

For a matrix g in the group $O(3)$, show that $\det g = \pm 1$.

Solution:

(a) The definition says $g^{-1} = g^T$ and $\det g = 1$.

Assume $g \in SO(3)$. Then also $g^T = g^{-1} \in SO(3)$: if $g^T g = \mathbb{1}$ then also $(g^T)^T g^T = \mathbb{1}$. Furthermore $\det g^T = \det g = 1$.

Clearly $\mathbb{1} \in SO(3)$ and group multiplication is associative.

Finally, if $g, g' \in SO(3)$ we have

$$(gg')^{-1} = g'^{-1} g^{-1} = g'^T g^T = (gg')^T \tag{0.0.2}$$

and

$$\det gg' = \det g \det g' = 1\,. \tag{0.0.3}$$

(b) Let $\boldsymbol{v}' = g\boldsymbol{v}$. Then

$$\begin{aligned} \text{length}^2(\boldsymbol{v}') &= \boldsymbol{v}' \cdot \boldsymbol{v}' = v_i' v_i' = g_{ij} v_j g_{ik} v_k = v_j g^T_{ji} g_{ik} v_k = \boldsymbol{v}^T g^T g \boldsymbol{v} = \boldsymbol{v} \cdot \boldsymbol{v} \\ &= \text{length}^2(\boldsymbol{v}) \end{aligned} \tag{0.0.4}$$

(c) We have $g^T = (e^\gamma)^T = e^{\gamma^T} = g^{-1} = e^{-\gamma}$. Hence $\gamma^T = -\gamma$. Repeating the same steps as done in the proof given in the lecture for $SU(2)$ shows that

$$1 = \det g = \det e^\gamma = e^{\mathrm{tr}\gamma} \tag{0.0.5}$$

so that we need the trace of γ to vanish.

(d) Recall $O(3)$ is the group of 3×3 matrices with $g^{-1} = g^T$. Using $g^T g = \mathbb{1}$ we have that $1 = \det \mathbb{1} = \det g^T g = \det g^T \det g = (\det g)^2$.

This implies that $SO(3)$ has two disjoint components, one with $\det g = 1$ and one with $\det g = -1$. As the determinant is a continuous function of the components of the matrix, there is no way we there is a continuous path that takes us from matrices with $\det = -1$ to those of $\det = +1$. Hence $O(3)$ has two connected components. One of these (the one with the $+$) containes the identity and is a subgroup called $SO(3)$. The other one (the one with the $-$) does not contain the identity and is hence not a subgroup.

2) Decide if the following maps are group homomorphisms.

(a) For $g \in U(1)$, $f : g \to g^2 \in U(1)$.

(b) For $g \in SU(2)$, $f : g \to g^2 \in SU(2)$.

(c) For $g \in SU(2)$,

$$f : g \to \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \in SU(2) \times SU(2) \tag{0.0.6}$$

Here $SU(2) \times SU(2)$ is the group of block-diagonal matrices of the form

$$\begin{pmatrix} g & 0 \\ 0 & h \end{pmatrix} \tag{0.0.7}$$

with $g \in SU(2)$ and $h \in SU(2)$.

Solution:

In each case, we need to check the homomorphism property. As all of these are multi-

plicative groups, we need to show $f(gg') = f(g)f(g')$.

(a) We have $f(gg') = (gg')^2 = gg'gg' = g^2(g')^2 = f(g)f(g')$. So this is a homomorphism as $U(1)$ is an abelian group.

(b) We have $f(gg') = (gg')^2 = gg'gg'$ but now this is not always equal to $g^2(g')^2 = f(g)f(g')$ as $gg' \neq g'g$ in general. So this is **not** a homomorphism as $SU(2)$ is a non-abelian group.

(c)

$$f(gg') = \begin{pmatrix} gg' & 0 \\ 0 & gg' \end{pmatrix} = \begin{pmatrix} g & 0 \\ 0 & g \end{pmatrix} \begin{pmatrix} g' & 0 \\ 0 & g' \end{pmatrix} = f(g)f(g') \tag{0.0.8}$$

so this is a homomorphism.