# Problem Class 2

***Problem 1:*** Show that the tangent vectors $T_p(S)$ at a point $\boldsymbol{p}$ form a real $n$-dimensional vector space.

**solution:**

To show this, we need to check that (i) both multiples $c\, T(S)$ for $v \in \mathbb{R}$ and (ii) sums $T(S) + T(S')$ also satisfy our definition of what a tangent vector is. Finally, we need to show that (iii) this vector space has a basis of dimension $n$.

i) Given a path $S : \boldsymbol{q}(t)$ that defines a tangent vector $T_p(S) = \partial \boldsymbol{q}(t)/\partial t|_{t_0}$ at $\boldsymbol{p}$, we can take $t_0 = 0$ without imposing any restriction and consider the path $S_c$ defined by $\boldsymbol{x}(ct)$. This path also runs through $p$ and the components of its tangent vector are

$$T_p(S_c) = \left.\frac{\partial \boldsymbol{q}'(ct)}{\partial t}\right|_0 = c \left.\frac{\partial \boldsymbol{q}'(t)}{\partial t}\right|_0 = cT_p(S). \tag{0.1}$$

For any tangent vector, there is hence another one with components that are a rescaled by a real number $c$.

ii) Here we want to add paths, which is not something we can easily do as for $\boldsymbol{q}(t)$ and $\boldsymbol{q}'(t)$ in $X$, their sum does not need to be in $X$. We can however pass to local coordinates using some coordinate map $\phi$ where these paths are given by $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$. In other words we have that

$$\begin{aligned} \phi(\boldsymbol{x}(t)) &= q(t) \\ \phi(\boldsymbol{x}'(t)) &= q'(t) \end{aligned} \tag{0.2}$$

Now, we can form a path $S''$ in local coords by

$$\boldsymbol{x}''(t) = \frac{1}{2}\left(\boldsymbol{x}(2t) + \boldsymbol{x}'(2t)\right) \tag{0.3}$$

which gives

$$\phi(\boldsymbol{x}''(t)) = q''(t)\,. \tag{0.4}$$

Let $\phi(\boldsymbol{x}_0) = \boldsymbol{p}$ As $\boldsymbol{x}(t)$ and $\boldsymbol{x}'(t)$ both pass through $x_0$, $\boldsymbol{x}''(t)$ does so as well. At $x_0$ we can compute

$$\begin{aligned} T_p(S'') &= \left.\frac{\partial \boldsymbol{q}''(t)}{\partial t}\right|_{\boldsymbol{p}} = \frac{\partial \phi}{\partial x''}\left.\frac{\partial \boldsymbol{x}''(t)}{\partial t}\right|_{\boldsymbol{x}_0} = \frac{\partial \phi}{\partial x''}\frac{1}{2}\left(\left.\frac{\partial x_i(2t)}{\partial t}\right|_{\boldsymbol{x}_0} + \left.\frac{\partial x_i'(2t)}{\partial t}\right|_{\boldsymbol{x}_0}\right) \\ &= \frac{\partial \phi}{\partial x''}\left(\left.\frac{\partial x_i(t)}{\partial t}\right|_{\boldsymbol{x}_0} + \left.\frac{\partial x_i'(t)}{\partial t}\right|_{\boldsymbol{x}_0}\right) = \left.\frac{\partial \boldsymbol{q}(t)}{\partial t}\right|_{\boldsymbol{p}} + \left.\frac{\partial \boldsymbol{q}'(t)}{\partial t}\right|_{\boldsymbol{p}} = T_p(S) + T_p(S') \end{aligned} \tag{0.5}$$

iii) To see this, first note that we can choose paths that have $x_i(t) = t$ and all other components vanishing. For such paths

$$\left.\frac{\partial \boldsymbol{x}(t)}{\partial t}\right|_{x_0} = (0, \cdots, 0, 1, 0, \cdots, 0)\,, \tag{0.6}$$

with an entry only at the $i$th component. In local coordinates, we can express any path as a linear combintation of those and there are $n$ linearly independent elements. As adding paths in local coordinates implies adding paths $\boldsymbol{q}(t)$, the corresponding linear relations hold for the tangent vectors.