# Problem Class 2

***Problem 1:*** The action

$$S = \int dt d^3x \quad -|\boldsymbol{\nabla}\psi|^2 + \tfrac{1}{2}i\left(\bar{\psi}\partial_t\psi - \psi\partial_t\bar{\psi}\right) \tag{0.1}$$

has a $U(1)$ symmetry $\psi \to e^{i\theta}\psi$, $\bar{\psi} \to e^{-i\theta}\bar{\psi}$, $\theta \in \mathbb{R}$. Find the associated conserved current.

**solution:**

First note that we should treat $\psi$ and $\bar{\psi}$ as independent fields. There is only a single Lie algebra element $\gamma = i\theta$ to consider and

$$\delta_\gamma\psi = i\theta\psi \qquad \delta_\gamma\bar{\psi} = -i\theta\bar{\psi} \tag{0.2}$$

and

$$\begin{aligned} j^0 &= i\theta\psi\frac{\partial\mathcal{L}}{\partial\partial_t\psi} - i\theta\bar{\psi}\frac{\partial\mathcal{L}}{\partial\partial_t\bar{\psi}} = -|\psi|^2 \\ j^j &= i\theta\psi\frac{\partial\mathcal{L}}{\partial\partial_j\psi} - i\theta\bar{\psi}\frac{\partial\mathcal{L}}{\partial\partial_j\bar{\psi}} \\ &= i\theta\psi(-\partial_j\bar{\psi}) - i\theta\bar{\psi}(-\partial_j\psi) \\ &= -i\theta\left(\psi\partial_j\bar{\psi} - \bar{\psi}\partial_j\psi\right) \end{aligned} \tag{0.3}$$

and the conservation equation is

$$\partial_t|\psi|^2 + \partial_j\left(i\psi\partial_j\bar{\psi} - i\bar{\psi}\partial_j\psi\right) = 0 \tag{0.4}$$

This equation guarantees that probability is conserved in quantum mechanics:

$$\frac{\partial}{\partial t}\int_{\mathbb{R}^3} d^3x|\psi|^2 = 0\,, \tag{0.5}$$

and what we have just seen is that this is enforced by a $U(1)$ symmetry!

***Problem 2:*** Consider the following action of a complex scalar field:

$$\mathcal{L} = \partial_\mu\bar{\phi}\partial^\mu\phi + m^2|\phi|^2\,. \tag{0.6}$$

a) Find the conserved current $j^\mu$ associated to a $U(1)$ symmetry of $\mathcal{L}$ acting as $\phi \to e^{i\theta}\phi$, $\theta \in \mathbb{R}$.

b) Show that $j^\mu$ is real. Is $j^0$ always positive? [hint: try plane wave solutions $\phi = \exp\{ik_\mu x^\mu\}$].

**solution:**

a) We have

$$j^\mu = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)}\delta_\gamma \phi + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})}\delta_\gamma \bar{\phi} \tag{0.7}$$

Furthermore

$$\delta_\gamma \phi = i\theta\phi \qquad \delta_\gamma \bar{\phi} = -i\theta\bar{\phi}\,. \tag{0.8}$$

We can write

$$\mathcal{L} = \eta^{\rho\sigma}\partial_\rho \bar{\phi}\partial_\sigma \phi + m^2|\phi|^2\,. \tag{0.9}$$

so that

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi)} = \frac{\partial \eta^{\rho\sigma}\partial_\rho \bar{\phi}\partial_\sigma \phi}{\partial(\partial_\mu \phi)} = \partial^\mu \bar{\phi} \tag{0.10}$$

and

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \bar{\phi})} = \frac{\partial \eta^{\rho\sigma}\partial_\rho \bar{\phi}\partial_\sigma \phi}{\partial(\partial_\mu \bar{\phi})} = \partial^\mu \phi \tag{0.11}$$

Hence we have all the ingredients in place to write down

$$j^\mu = (\partial^\mu \bar{\phi})i\theta\phi + (\partial^\mu \phi)(-i\theta\bar{\phi}) = i\theta\left(\phi\partial^\mu\bar{\phi} - \bar{\phi}\partial^\mu\phi\right) \tag{0.12}$$

Of course this is conserved for any $\theta$, so that we might as well write this as

$$j^\mu = i\left(\bar{\phi}\partial^\mu\phi - \phi\partial^\mu\bar{\phi}\right) \tag{0.13}$$

b) We work out

$$\bar{j}^\mu = -i\left(\phi\partial^\mu\bar{\phi} - \bar{\phi}\partial^\mu\phi\right) = j^\mu \tag{0.14}$$

so that the current must be real. A plane wave solution might be

$$\phi(\boldsymbol{x}) = \exp\left(ik_\mu x^\mu\right)\,. \tag{0.15}$$

Plugging this into the equations of motion gives

$$\begin{aligned} 0 &= (\partial_\mu\partial^\mu - m^2)\exp\left(ik_\nu x^\nu\right) = (\eta^{\rho\sigma}\frac{\partial}{\partial x^\rho}\frac{\partial}{\partial x^\sigma} - m^2)\exp\left(ik_\nu x^\nu\right) \\ &= (-\eta^{\rho\sigma}k_\rho k_\sigma - m^2)\exp\left(ik^\nu x_\nu\right) = (-k_\nu k^\nu - m^2)\exp\left(ik^\nu x_\nu\right) \end{aligned} \tag{0.16}$$

so that we need to choose $k_\nu$ such that $k^\nu k_\nu = -k_0^2 + k_1^2 + k_2^2 + k_3^2 = -m^2$. Note that we could e.g. have solutions with $k_0 = \pm m$ and $k_i = 0$ for $i = 1, 2, 3$. We then have that

$$j^0 = -2k^0 \tag{0.17}$$

and depending on the sign of $k^0$ this is either positive or negative. If we wanted to use $\phi$ as a wave-function we would then be in trouble when trying to interpret $j^0$ as a probability density. This is why the Klein-Gordon equation could not be used to describe relativistic quantum mechanics using wave-functions.