Problem Class 3

Problem 1: Show that the Lie algebra of $SO(3)$ has the same structure constants as the Lie algebra of $SU(2)$ in an appropriate basis.

solution:

We already worked out the tangent space of $SO(3)$ in the problem sheet for week 5. For the path

$$s_1(t) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(t) & \sin(t) \\ 0 & -\sin(t) & \cos(t) \end{pmatrix} \tag{0.1}$$

in $SO(3)$ we cross $\mathbb{1}$ for $t = 0$. We can compute the associated tangent vector

$$\ell_1 \equiv T_{\mathbb{1}}(s) = \frac{\partial}{\partial t} s(t)|_{t=0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \tag{0.2}$$

After permuting the different directions we also get the Lie algebra elements from the corresponding paths.

$$\ell_2 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \qquad \ell_3 = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.3}$$

We also showed that for $g \in SO(3)$ we can write

$$g = e^{\gamma} \tag{0.4}$$

with γ a real matrix s.t. $\gamma^T = -\gamma$. We can hence find the same result by considering the path

$$s_\gamma(t) = e^{t\gamma} \tag{0.5}$$

for any γ with $\gamma^T = -\gamma$. We compute

$$T_{\mathbb{1}}(s_\gamma) = \frac{\partial}{\partial t} s_\gamma(t)|_{t=0} = \gamma . \tag{0.6}$$

The three matrices above are the most general matrices which obey $\gamma^T = -\gamma$.

A direct computation shows that they satisfy

$$[\ell_i, \ell_j] = \epsilon_{ijk} \ell_k \,. \tag{0.7}$$

Recall that the Lie algebra of $SU(2)$ has basis vectors $i\sigma_k$ for $k = 1, 2, 3$ and the σ_k obey

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \,. \tag{0.8}$$

which is the same except for the factor of $2i$. Letting $\sigma_k = 2i\hat{\sigma}_k$ we get

$$[\hat{\sigma}_i, \hat{\sigma}_j] = \epsilon_{ijk}\hat{\sigma}_k \,. \tag{0.9}$$

which is the same algebra as the one of $SO(3)$. Note that it does not matter which matrices we write ! The whole structure is that of a vector space and map $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, the commutator $[.,.]$, which can be summarized by structure constants for any given basis.

What we have shown here is that both Lie algebras real 3-dimensional, and we can choose a basis where the structure constants are the same.

On the one hand, this result might not be unexpected as $SU(2)$ and $SO(3)$ are 'the same' in the vicinity of the identity: recall there is a 2-1 map from $SU(2)$ to $SO(3)$ which send $g, -g$ to the same element in $SO(3)$. To work out the Lie algebra we restrict ourselves to a small open set containing $\mathbb{1}_{SU(2)}$, which is then mapped bijectively to a small open set containing $\mathbb{1}_{SO(3)}$.

On the other hand, you might find it unsettling that 'the same' Lie algebra can give different groups. More concretely, one way to see what is going on is to observe that we get $SU(2)$ when exponentiating (i times the) Pauli matrices, whereas we get elements $SO(3)$ when exponentiating the matrices found above. It hence matters how we 'represent' this abstract thing that is a Lie algebra. (we will properly define representations of Lie algebras later).

Problem 2: Consider the set G of matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{R}, ac \neq 0 \right\} \tag{0.10}$$

1. Show that G is a Lie group using matrix multiplication as the group composition.
2. Find the Lie algebra $\mathfrak{g}$ of G.
3. Compute the exponentials of the basis elements of the Lie algebra you have found.

solution:

1. We compute

$$\begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \begin{pmatrix} a' & b' \\ 0 & c' \end{pmatrix} = \begin{pmatrix} aa' & ab' + bc' \\ 0 & cc' \end{pmatrix} \tag{0.11}$$

so the product of any two of these is also in G. Also we can check that

$$g^{-1} = \frac{1}{ac}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \tag{0.12}$$

is in G for any $g \in G$.

It is furthermore a subgroup of $GL(2,\mathbb{R})$ that is closed, so that it must be a Lie group. Here are more details: writing a general matrix in $GL(2,\mathbb{R})$ as

$$\begin{pmatrix} a & b \\ d & c \end{pmatrix} \tag{0.13}$$

with $ac - bd \neq 0$, we can characterize G by $d = 0$. This is a closed condition, for any $d \neq 0$ we can find a little open ball for which still $d \neq 0$.

2. We need define an appropriate number of paths in G. As G is real three-dimensional, the tangent space at $\mathbb{1}$ is a three-dimensional vector space. We can write every element of G as

$$g = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} = \begin{pmatrix} e^x & y \\ 0 & e^z \end{pmatrix} \tag{0.14}$$

as $a \neq 0$ and $c \neq 0$. Here $(x, y, z) \in \mathbb{R}^3$. We can hence write some paths in G as

$$\begin{pmatrix} e^{xt} & yt \\ 0 & e^{zt} \end{pmatrix} \tag{0.15}$$

which all go through $g = \mathbb{1}$ at $t = 0$. Other ways of writing paths are fine too of course, but the above is the most convenient. Now we can work out

$$\frac{\partial}{\partial t}\begin{pmatrix} e^{xt} & yt \\ 0 & e^{zt} \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} x & y \\ 0 & z \end{pmatrix} . \tag{0.16}$$

The set of these matrices is isomorphic to $\mathbb{R}^3$ as $(x, y, z) \in \mathbb{R}^3$. As we already know that G is real three-dimensional this spans the whole Lie algebra of G. Note that these might look the same as the elements of G but, now x, y are allowed to vanish (after all, this is a vector space which must contain the zero vector).

3. This is best worked out by using the definition of the exponential

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!} \tag{0.17}$$

$$\exp\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix} = \sum_k \frac{\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}^k}{k!} = \begin{pmatrix} \sum_{k=0}^\infty \frac{1}{k!} x^k & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^x & 0 \\ 0 & 1 \end{pmatrix} \tag{0.18}$$

We have used that

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}^0 = \mathbb{1} \tag{0.19}$$

and

$$\begin{pmatrix} x & 0 \\ 0 & 0 \end{pmatrix}^k = \begin{pmatrix} x^k & 0 \\ 0 & 0 \end{pmatrix} \tag{0.20}$$

for $k > 0$.
Similarly

$$\exp\begin{pmatrix} 0 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & e^z \end{pmatrix} \tag{0.21}$$

Finally

$$\exp\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix} + \sum_{k=2}^\infty \frac{1}{k!} \begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}^k = \begin{pmatrix} 1 & y \\ 0 & 1 \end{pmatrix} \tag{0.22}$$

using that powers of two or higher of $\begin{pmatrix} 0 & y \\ 0 & 0 \end{pmatrix}$ vanish.