Problem Class 3

Problem 1: In section 5.3. we have assumed for simplicity that ϕ transforms as

$$\phi \to e^{i\alpha}\phi$$

under a global $U(1)$ symmetry and then gauged this symmetry. Assume that ϕ is acted on in a different complex irreducible representation of $U(1)$ and adjust all equations in section 5.3. accordingly.

solution:

First we need to recall what complex irreducible representations of $U(1)$ are like: parametrizing $U(1)$ by $e^{i\alpha}$, they are given by $r_k : e^{i\alpha} \mapsto e^{ik\alpha}$ for $k \in \mathbb{Z}$. A field $\hat{\phi}$ transforming under a gauge transformation associated with r_k would then transform as

$$\hat{\phi} \to e^{ik\alpha(x)}\hat{\phi}\,, \tag{0.1}$$

while the gauge field A_μ still behaves as

$$A_\mu \to e^{i\alpha}\left(A_\mu + \partial_\mu\right)e^{-i\alpha} = A_\mu + \partial_\mu\alpha \tag{0.2}$$

The key point in the construction of gauge invariant dynamics is 5.3. was the covariant derivative

$$D_\mu\phi = \partial_\mu\phi - iA_\mu\phi \tag{0.3}$$

which had the property that

$$D_\mu\phi \mapsto D'_\mu\phi' = e^{i\alpha}D_\mu\phi\,, \tag{0.4}$$

i.e. it transforms the same way as ϕ. Hence we now want that

$$D_\mu\hat{\phi} \to e^{ik\alpha}D_\mu\hat{\phi}\,. \tag{0.5}$$

The construction of the covariant derivative was motivated by cancelling the unwanted derivative of α by the shift, and we can do the same thing here with a little tweak by defining

$$D_\mu\hat{\phi} := \partial_\mu\hat{\phi} - ikA_\mu\hat{\phi}\,. \tag{0.6}$$

Let us check this does what it should:

$$\begin{aligned} D_\mu\hat{\phi} \mapsto D'_\mu\hat{\phi}' &= \partial_\mu\left(e^{ik\alpha}\hat{\phi}\right) - ik(A_\mu + (\partial_\mu\alpha))e^{ik\alpha(x)}\hat{\phi} \\ &= e^{ik\alpha}\partial_\mu\hat{\phi} + ike^{ik\alpha}\hat{\phi}\partial_\mu\alpha - iA_\mu ke^{ik\alpha}\hat{\phi} - ike^{ik\alpha}\hat{\phi}\partial_\mu\alpha \\ &= e^{ik\alpha}\left(\partial_\mu\hat{\phi} - ikA_\mu\hat{\phi}\right) = e^{ik\alpha}D_\mu\hat{\phi} \end{aligned} \tag{0.7}$$

All we need to do is hence to use the covariant derivative $D_\mu = \partial_\mu \hat{\phi} - ikA_\mu \hat{\phi}$ instead thoughout 5.3 and we are done. It is common to still write D_μ in the understanding that a covariant derivative acts on a field depending on its transformation behavior.

Note that the current j_μ and hence the coupling of ϕ to A_μ gets rescaled by a factor of k, which can even be negative. For this reason k is called the charge of the field $\hat{\phi}$.

Problem 2: We have seen the Schroedinger action

$$S = \int dt d^3x - \boldsymbol{\nabla}\psi \cdot \overline{\boldsymbol{\nabla}\psi} + i\tfrac{1}{2}\left(\bar{\psi}\partial_t\psi - \psi\partial_t\bar{\psi}\right) \tag{0.8}$$

in the lectures which gave Schroedinger's equation as the equation of motion of a classical field theory, and observed that it has a global $U(1)$ symmetry

$$\psi \to e^{i\alpha}\psi \tag{0.9}$$

which guaranteed conservation of the charge $Q = \int d^3x|\psi|^2$ interpreted as probability conservation in quantum mechanics.

Turn this $U(1)$ into a gauge symmetry by letting $\alpha = \alpha(t, x)$ and derive the equations of motion of ψ (we have made the dependence of α on both time and space explicit, i.e. are using a non-relativistic notation here).

solution:

Again it is the derivatives which are an issue, but we can solve this in the same way as done for a relativistic scalar. We simply replace

$$\boldsymbol{\nabla} \to \boldsymbol{D} := \boldsymbol{\nabla} - i\boldsymbol{A} \qquad \partial_t \to D_t := \partial_t - i\phi \tag{0.10}$$

where $A_0 = \phi$ and $\boldsymbol{A} = A_1, A_2, A_3$ are the usual electrostatic and vector potential appearing in a non-relativistic formulation of Maxwell's equations. With these replacements the gauged Schroedinger action reads (omitting a kinetic term for the gauge field A_μ):

$$\begin{aligned} S &= \int dt d^3x - \boldsymbol{D}\psi \cdot \overline{\boldsymbol{D}\psi} + i\tfrac{1}{2}\left(\bar{\psi}D_t\psi - \psi\overline{D_t\psi}\right) \\ &= \int dt d^3x - ((\boldsymbol{\nabla} - i\boldsymbol{A})\psi) \cdot \left((\boldsymbol{\nabla} + i\boldsymbol{A})\overline{\psi}\right) + i\tfrac{1}{2}\left(\bar{\psi}(\partial_t - i\phi)\psi - \psi(\partial_t + i\phi)\bar{\psi}\right) \end{aligned} \tag{0.11}$$

The Euler-Lagrange equations for ψ are the complex conjugates of those of $\bar{\psi}$, and to get an equation for ψ we work out those. To write down the Euler-Lagrange equations for $\bar{\psi}$, we work out

$$\begin{aligned} \frac{\partial}{\partial\overline{\psi}}\mathcal{L} &= \phi\psi + \frac{1}{2}i\partial_t\psi - i\boldsymbol{A}\cdot(\boldsymbol{\nabla} - i\boldsymbol{A})\psi \\ \frac{\partial}{\partial\partial_t\overline{\psi}}\mathcal{L} &= -\frac{1}{2}i\psi \\ \frac{\partial}{\partial\partial_j\overline{\psi}}\mathcal{L} &= -\left((\boldsymbol{\nabla} - i\boldsymbol{A})\psi\right)_j \end{aligned} \tag{0.12}$$

where $\partial_j = \partial/\partial x_j$, $j = 1, 2, 3$.

Hence the Euler Lagrange equation

$$\frac{\partial}{\partial\overline{\psi}}\mathcal{L} - \partial_t\frac{\partial}{\partial\partial_t\overline{\psi}}\mathcal{L} - \partial_j\frac{\partial}{\partial\partial_j\overline{\psi}}\mathcal{L} = 0 \tag{0.13}$$

gives

$$\begin{aligned} 0 &= \phi\psi + \frac{1}{2}i\partial_t\psi - \boldsymbol{A}\cdot(\boldsymbol{\nabla} - i\boldsymbol{A})\psi + \frac{1}{2}i\partial_t\psi + \boldsymbol{\nabla}(\boldsymbol{\nabla} - i\boldsymbol{A})\psi \\ &= iD_t\psi + \boldsymbol{D}\boldsymbol{D}\psi \end{aligned} \tag{0.14}$$

To no surpise the e.o.m. contains covariant derivates only and is gauge covariant.

Note that we can rewrite this as

$$\boldsymbol{D}\boldsymbol{D}\psi + \phi\psi = -i\partial_t\psi \tag{0.15}$$

which means that in QM we would use

$$\hat{H} = -(\boldsymbol{\nabla} - i\boldsymbol{A})^2 - \phi \tag{0.16}$$

as the Hamilton operator. This is just the quantum version of the Hamiltonian of a charged particle of mass 1/2 and charge −1 in an electro-magnetic field.

Problem 3: Repeat problem 2 for the Dirac action

$$S = \int d^4x\bar{\Psi}\left(\gamma^\mu\partial_\mu + m\right)\Psi \tag{0.17}$$

Here we can use the same principle and replace $\partial_\mu \to D_\mu := \partial_\mu - iA_\mu$. resulting in (again ignoring the kinetic term for the gauge field A_μ):

$$S = \int d^4x\bar{\Psi}\left(\gamma^\mu D_\mu + m\right)\Psi \tag{0.18}$$

The e.o.m is simply

$$\left(\gamma^\mu D_\mu + m\right)\Psi = 0\,. \tag{0.19}$$

This is the Dirac equation describing a charged electron in an electro-magnetic field and can be used to find the celebrated result that the magnetic moment of an electron (or rather the so-called g-factor) is 2. Combining the ideas of the spin-$\frac{1}{2}$ 'representation' of the Lorentz group and gauge invariance forces us to write down the above version of the Dirac equation, which in turn explains and experimental result which had been a mystery before.