

## Problem Class 4

**Problem 1:** The group  $SL(2, \mathbb{C})$  is the group of complex invertible  $2 \times 2$  matrices  $g$  with  $\det g = 1$

1. Find the Lie algebra of  $\mathfrak{sl}(2, \mathbb{C})$ .
2. Show that the complex irreducible representations of  $\mathfrak{sl}(2, \mathbb{C})$  are equivalent to the irreducible representations of  $\mathfrak{su}(2)$ .
3. Show that any irreducible representation of  $SU(2)$  also gives rise to a irreducible representation of  $\mathfrak{su}(2)$ .

**solution:**

1. Note that writing

$$g = e^{i\alpha_j \sigma_j} \tag{0.1}$$

gives us an element in  $\mathfrak{sl}(2, \mathbb{C})$  for any  $\alpha \in \mathbb{C}^3$ . These are all invertible and have  $\det g = 1$ , and are the most general such matrices you can write. We can then write down a path

$$g = e^{it\alpha_j \sigma_j} \quad \text{for } t = -a \dots a \tag{0.2}$$

which tells us that

$$\mathfrak{sl}(2, \mathbb{C}) \supset \{i\alpha_j \sigma_j | \alpha \in \mathbb{C}^3\}. \tag{0.3}$$

Note that the dimension of  $SL(2, \mathbb{C})$  as a manifold is 6: a complex  $2 \times 2$  matrix has  $4 \times 2$  real entries and  $\det g = 1$  are two real conditions on those (note that  $\det g \in \mathbb{C}$ ). As the set above is also a real vector space of dimension 6, it must be the whole of the Lie algebra.

We can express the commutators of elements of this vector space in terms of the basis vectors, which are the Pauli matrices  $\sigma_j$ :

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k \tag{0.4}$$

so we are done.

2. As a representation maps elements of the Lie algebra linearly to matrices  $\in \mathfrak{gl}(V)$ , we can write

$$\rho(\gamma) = \rho(it\alpha\sigma) = \sum_j i\alpha_j \rho(\sigma_j) \tag{0.5}$$

i.e. we only need to know how the Pauli matrices are represented for both cases. This means we can go back and forth between representations of these two groups by changing if we let the  $\alpha_i$  be real or complex.

As we have complex representations, irreducibility means there should not be non-trivial invariant **complex** subspaces.

If there is an invariant subspace  $W$  for any of the two cases, then  $\rho(\sigma_i)\mathbf{w} \in W$  for all  $w \in W$  as a special case. Conversely, if  $\rho(\sigma_i)\mathbf{w} \in W$  for all  $w \in W$ , then  $\sum_j i\alpha_j \rho(\sigma_j)\mathbf{w} \in W$  for all  $w \in W$ . Hence the condition for (ir-)reducibility is the same.

3. We can prove this by a contradiction. Assume that  $r$  is irreducible and assume that  $\rho$  is reducible. If  $\rho$  is a reducible representation, there is an invariant subspace  $\rho(\gamma)w \in W \forall w \in W$  and  $\forall \gamma \in \mathfrak{su}(2)$ . This implies that  $W$  is also an invariant subspace for

$$(\mathbb{1} + \rho(\gamma)/n)^n. \quad (0.6)$$

for any  $n$ . If we now take the limit  $n \rightarrow \infty$  we can conclude that

$$\lim_{n \rightarrow \infty} (\mathbb{1} + \rho(\gamma)/n)^n w \in W \quad (0.7)$$

for all  $w \in W$  as the limit of this series must end up in  $W$  as well if all terms are in  $W$ . This implies that

$$\lim_{n \rightarrow \infty} (\mathbb{1} + \rho(\gamma)/n)^n = e^\rho(\gamma) = r(e^\gamma) \quad (0.8)$$

has an invariant subspace for all  $\gamma$ . As  $\exp$  is surjective for  $SU(2)$ , we get all group elements this way, so that the representation  $r$  is reducible. This is a contradiction.

**Problem 2:** Let

$$P(\mathbf{z}) = \sum_{k=0}^d a_k z_1^k z_2^{d-k}. \quad (0.9)$$

be a complex homogeneous polynomial of degree  $d$  and call the complex  $d+1$ -dimensional vector space of all such polynomials (for a fixed  $d$ )  $\Pi_d$ . The map

$$r_d(g)P := P(g^{-1}\mathbf{z}). \quad (0.10)$$

where  $g^{-1} \in SU(2)$  acts on  $\mathbf{z} = (z_1, z_2)$  as

$$\begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \rightarrow g^{-1} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \quad (0.11)$$

defines a representation  $r_d$  of  $SU(2)$  acting on  $\mathbb{C}^{d+1} = \Pi_d$ . Find the Lie algebra representation  $\rho_d$  associated with this representation.

**solution:**

To work this out, let's first make a strategy. We need to compute

$$\left. \frac{\partial}{\partial t} r_d(g(t)) \right|_{t=0} \quad (0.12)$$

for  $g(t)$  a path in  $SU(2)$ . We can make the simple choice of looking at  $e^{it\sigma_j}$ . We didn't write down  $r_d(g)$  as a matrix, so let's just see how it acts on a given basis vector. I.e. we can work this out for each  $j$  by looking at what happens to each monomial (= basis vector), so let's choose  $z_1^k z_2^{d-k}$ .

First consider

$$g(t) = e^{it\sigma_3} = e^{it \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}} \quad (0.13)$$

acting on  $z_1^k z_2^{d-k}$ . The Pauli matrix in the exponential maps (note the signs originating from using  $g^{-1}$ )

$$\begin{aligned} z_1 &\rightarrow e^{-it} z_1 \\ z_2 &\rightarrow e^{it} z_2 \end{aligned} \quad (0.14)$$

Hence

$$\left. \frac{\partial}{\partial t} r(g(t)) z_1^k z_2^{d-k} \right|_{t=0} = \left. \frac{\partial}{\partial t} z_1^k z_2^{d-k} e^{it(d-2k)} \right|_{t=0} = i(d-2k) z_1^k z_2^{d-k}. \quad (0.15)$$

The Lie algebra representation  $\rho_d(i\sigma_3)$  of  $SU(2)$  rescales every basis vector by  $i(d-2k)$ .

Now we can work out a more complicated case.

$$g(t) = e^{it\sigma_1} = e^{it \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}} \quad (0.16)$$

acting on  $z_1^k z_2^{d-k}$ . Think of writing out this monstrosity as a power series in  $t$ . Anything quadratic or higher order in  $t$  for every factor will give zero at the end of the day no matter what it multiplies. Hence it is good enough to write out the action to linear order in  $t$ :

$$\begin{aligned} z_1 &\rightarrow z_1 - itz_2 \\ z_2 &\rightarrow z_2 - itz_1 \end{aligned} \quad (0.17)$$

So

$$\begin{aligned} \left. \frac{\partial}{\partial t} r(g(t)) z_1^k z_2^{d-k} \right|_{t=0} &= \left. \frac{\partial}{\partial t} (z_1 - itz_2)^k (z_2 - itz_1)^{d-k} \right|_{t=0} \\ &= -ikz_1^{k-1} z_2^{d-k+1} - i(d-k)z_1^{k+1} z_2^{d-k-1}. \end{aligned} \quad (0.18)$$

All we did here is to collect all the terms linear in  $t$ . The map  $\rho(i\sigma_1)$  is

$$z_1^k z_2^{d-k} \rightarrow -ik z_1^{k-1} z_2^{d-k+1} - i(d-k) z_1^{k+1} z_2^{d-k-1} \quad (0.19)$$

You can work out  $\rho(\sigma_2)$  analogously or by using

$$\rho(\sigma_2) = \rho(\tfrac{i}{2}[\sigma_1, \sigma_3]) = \tfrac{i}{2}[\rho(\sigma_1), \rho(\sigma_3)] \quad (0.20)$$