

Problem Class 4

Problem 1: Write down the most general real gauge invariant and Lorentz invariant Lagrangian with at most two derivatives for two complex scalar fields, ϕ of charge 1 and χ of charge 2, and a $U(1)$ gauge field A_μ which contains

1. standard kinetic terms for ϕ and χ ;

solution:

Since the scalar fields are charged under a $U(1)$ gauge symmetry, we should replace the partial derivatives in the standard kinetic term $-\partial^\mu \bar{\phi} \partial_\mu \phi - \partial^\mu \bar{\chi} \partial_\mu \chi$ by gauge covariant derivatives. We need to remember that the gauge covariant derivative of a field of charge q is $D_\mu = \partial_\mu - iqA_\mu$. So we have the gauge invariant kinetic terms

$$\mathcal{L}_{\text{kin}} = -(\partial^\mu \bar{\phi} + iA^\mu \bar{\phi})(\partial_\mu \phi - iA_\mu \phi) - (\partial^\mu \bar{\chi} + 2iA^\mu \bar{\chi})(\partial_\mu \chi - 2iA_\mu \chi) .$$

2. a kinetic term for A_μ ;

solution:

This is simply the Maxwell Lagrangian density that we saw in the gauge theory formulation of electromagnetism:

$$\mathcal{L}_{\text{Maxwell}} = -\frac{1}{4g^2} F^{\mu\nu} F_{\mu\nu}$$

where $F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu$.

3. a real gauge invariant potential which is a polynomial of degree at most 4 in ϕ , χ and their complex conjugates.

solution:

We start by noticing that for any fields f_1, f_2 of charges q_1, q_2 respectively under a $U(1)$ symmetry (global or local/gauge), their product $f_1 f_2$ has charge $q_1 + q_2$. Indeed under a $U(1)$ transformation with group element $e^{i\alpha}$ the fields transform as

$$(f_1, f_2) \mapsto (e^{iq_1\alpha} f_1, e^{iq_2\alpha} f_2) \quad \implies \quad f_1 f_2 \mapsto e^{i(q_1+q_2)\alpha} f_1 f_2 .$$

This generalizes by induction to monomials in the fields: the charge of a monomial is the sum of the charges of its factors. In particular, a monomial in the fields is invariant under a $U(1)$ gauge transformation if and only if it has charge 0.

The scalar potential is a polynomial in $\phi, \chi, \bar{\phi}, \bar{\chi}$. Demanding gauge invariance (*i.e.* vanishing total charge), we see that the only allowed monomials are

$$1, \quad |\phi|^2 = \bar{\phi}\phi, \quad |\chi|^2 = \bar{\chi}\chi, \quad \bar{\chi}\phi^2, \quad \bar{\phi}^2\chi$$

and products/powers thereof. Therefore the most general real gauge invariant potential which is a polynomial of degree at most 4 in ϕ, χ and their complex conjugates is

$$V(\bar{\phi}, \bar{\chi}, \phi, \chi) = V_0 + m_\phi^2 |\phi|^2 + m_\chi^2 |\chi|^2 + \operatorname{Re}(a \bar{\chi} \phi^2) + \lambda_\phi |\phi|^4 + \lambda_\chi |\chi|^4 + \lambda_{\phi\chi} |\phi|^2 |\chi|^2,$$

where $V_0, m_\phi^2, m_\chi^2, \lambda_\phi, \lambda_\chi, \lambda_{\phi\chi}$ are real constants, and a is a complex constant. The constant V_0 (the ‘vacuum energy density’) is often ignored since it drops out of the equations of motion, and the energy is defined up to an additive constant.

Problem 2: Check by direct computation that

$$D_\mu \phi := (\partial - iA_\mu)\phi \rightarrow g(x)D_\mu \phi \tag{0.1}$$

for

$$\begin{aligned} \phi &\rightarrow g\phi \\ A_\mu &\rightarrow g(A_\mu + i\partial_\mu)g^{-1} \end{aligned} \tag{0.2}$$

solution: We work out

$$\begin{aligned} (\partial - iA_\mu)\phi &\rightarrow (\partial_\mu - ig(A_\mu g^{-1} + i\partial_\mu g^{-1}))g\phi \\ &= (\partial_\mu g)\phi + g\partial_\mu \phi - igA_\mu g^{-1}g\phi + (g\partial_\mu g^{-1})g\phi \\ &= gD_\mu \phi + (\partial_\mu g)\phi + (g\partial_\mu g^{-1})g\phi \end{aligned} \tag{0.3}$$

As $0 = \partial_\mu(gg^{-1}) = (\partial_\mu g)g^{-1} + g\partial_\mu g^{-1}$ the two last terms become

$$(\partial_\mu g)\phi - (\partial_\mu g)g^{-1}g\phi = 0 \tag{0.4}$$

and we can declare success!