1. Consider the representation $\mathbf{n} \otimes \bar{\mathbf{n}}$ of $SU(n)$. Explain why this is always reducible. Can you identify the irreducible representations and invariant subspaces?

   solution: We can repeat the same derivation as done in the lecture for $\mathbf{2} \otimes \bar{\mathbf{2}}$. Writing a general element A of $\mathbf{n} \otimes \bar{\mathbf{n}}$ as

   $$\sum_{ij} a_{ij} e_i \otimes f_j \tag{0.1}$$

   using basis vectors e_i and f_j for the vector space living in the $\mathbf{n}$ and $\bar{\mathbf{n}}$, we get the transformation law

   $$A \to gAg^\dagger \tag{0.2}$$

   for $g \in SU(n)$. The trace of A defines a subspace of the vector space spanned by all A, and we can work out

   $$trA \to trgAg^\dagger = trA \tag{0.3}$$

   so this is an invariant subspace $\mathbf{1}$. The orthogonal complement of this subspace has complex dimension $n^2 - 1$ and transforms in the same way as (a complex version of) the adjoint is hence irreducible. Hence we can write $\mathbf{n} \otimes \bar{\mathbf{n}} = \mathbf{1} \oplus \mathbf{adj}_{\mathbb{C}}$.

2. (a) Find the transformation of elements of $\mathbf{2} \otimes \mathbf{2}$.

   (b) Show that the representations $\mathbf{2}$ and $\bar{\mathbf{2}}$ are isomorphic by showing they are related by a change of basis

   $$\boldsymbol{z}' = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \boldsymbol{z} \tag{0.4}$$

   [Note: of course, $\bar{v}$ transforms also as $\bar{v} \to \bar{g}\bar{v}$ if $v \to gv$. In a complex vector space, complex conjugation is not a change of basis however!]

   (c) Use the above to argue that $\mathbf{2} \otimes \mathbf{2} = \mathbf{1} \oplus \mathbf{3}$. Can you identify the invariant subspaces?

   solution:

   (a) Using the general formula from the lecture, elements A of $\mathbf{2} \otimes \mathbf{2}$ written as a matrix transform as

   $$A \to gAg^T \,. \tag{0.5}$$

   (b) Let us assume that $\boldsymbol{z}$ lives in the $\mathbf{2}$, so that it transforms as

   $$\boldsymbol{z} \to g\boldsymbol{z} \,. \tag{0.6}$$

Define the coords after the change basis as

$$\begin{pmatrix} u_1 & u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \tag{0.7}$$

This means that the action of g on $\mathbb{C}^2$ after this basis change is

$$\begin{aligned} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} &\to \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} g \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}^{-1} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} \bar{a} & \bar{b} \\ -b & a \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \end{aligned} \tag{0.8}$$

Hence this change of basis takes us from an action by g to one by $\bar{g}$. Note how the specific form of g was important here.

(c) After this change of basis on one of the factors of the tensor product $\mathbb{C}^2 \otimes \mathbb{C}^2$ we are considering, the action is exactly as it was for the $\mathbf{2} \otimes \bar{\mathbf{2}}$! Then we are considering the same as the problem treated in the lecture (and above for the general case) and conclude that

$$\mathbf{2} \otimes \mathbf{2} \cong \mathbf{1} \oplus \mathbf{3} \tag{0.9}$$

again.

To explicitely find the singlet in the original version of writing things

$$A \to g A g^T \,. \tag{0.10}$$

we can proceed as follows: let $\epsilon_{12} = -\epsilon_{21}$. For $g \in SU(2)$ we can write

$$\det g = \epsilon_{ij} g_{i1} g_{j2} = 1 \tag{0.11}$$

or equivalently

$$\epsilon_{ij} g_{ik} g_{jl} = \epsilon_{kl} \tag{0.12}$$

Hence

$$\epsilon_{ij} a_{ij} \to \epsilon_{ij} g_{ik} g_{jl} a_{kl} = \epsilon_{kl} a_{kl} \tag{0.13}$$

is a singlet. You can find the same by mapping the trace of a matrix in $\mathbf{2} \otimes \bar{\mathbf{2}}$ back to $\mathbf{2} \otimes \mathbf{2}$.