4. Show using index notation that

   a) $(A+B)^T = A^T + B^T$
   b) $(AB)^T = B^T A^T$
   c) $tr(cA) = c\, tr(A)$
   d) $tr(AB) = tr(BA)$
   e) $tr A^T = tr A$
   f) $tr(A+B) = tr A + tr B$
   g) $(A\boldsymbol{v}) \cdot (B\boldsymbol{w}) = \boldsymbol{v}\,(A^T B)\,\boldsymbol{w}$
   h) $\det A^\dagger = \overline{\det A}$
   i) $\det cA = c^n \det A$

   where A and B are complex $n \times n$ matrices, $\boldsymbol{v}$ and $\boldsymbol{w}$ are vectors with n components, and c is a number.
   Solution:

   a) This is a matrix equation, to show it we just show it holds for all components, i.e. we start investigating $\left((A+B)^T\right)_{ij}$. So here we go:

   $$\left((A+B)^T\right)_{ij} = (A+B)_{ji} = A_{ji} + B_{ji} = \left(A^T\right)_{ij} + \left(B^T\right)_{ij} . \tag{0.1}$$

   where have used the definition of transposition written out in components and the fact that a sum of matrices sums their components.

   b)

   $$\begin{aligned}\left((AB)^T\right)_{ij} &= ((AB))_{ji} = A_{jk}B_{ki} = \left(A^T\right)_{kj}\left(B^T\right)_{ik} = \left(B^T\right)_{ik}\left(A^T\right)_{kj} \\ &= \left(B^T A^T\right)_{ij}\end{aligned} \tag{0.2}$$

   Here we have used that we can freely commute components of matrices (unlike matrices themselves) because these are just numbers. If it makes you uncomfortable, you can reinstate the summation signs which we are suppressing as we are using summation convention. Again, we have show that $(AB)^T = B^T A^T$ by showing that all components of the expressions on both sides agree.

   c)

   $$trcA = (cA)_{ii} = c\,(A)_{ii} = c\; trA\,. \tag{0.3}$$

   Here we used that multiplying a matrix by a number just multiplies its components.

d) $tr(AB) = A_{ij}B_{ji} = B_{ji}A_{ij} = tr(BA)$

e) $trA^T = \left(A^T\right)_{ii} = A_{ii} = trA$

f) $tr(A+B) = (A+B)_{ii} = A_{ii} + B_{ii} = trA + trB$

g)

$$\begin{aligned}(A\boldsymbol{v})\cdot(B\boldsymbol{w}) =& A_{ij}v_jB_{ik}w_k = v_j\left(A^T\right)_{ji}B_{ik}w_k = v_j\left(A^TB\right)_{jk}w_k \\ =& \boldsymbol{v}\left(A^TB\right)\boldsymbol{w}\end{aligned} \tag{0.4}$$

h)

$$\begin{aligned}\det A^\dagger =& \epsilon_{i_1,i_2,\cdots i_n}\left(A^\dagger\right)_{1i_1}\left(A^\dagger\right)_{2i_2}\cdots\left(A^\dagger\right)_{ni_n} = \overline{\epsilon_{i_1,i_2,\cdots i_n}A_{i_11}A_{i_22}\cdots A_{i_nn}} \\ =& \overline{\det A^T} = \overline{\det A}\end{aligned} \tag{0.5}$$

i)

$$\begin{aligned}\det cA =& \epsilon_{i_1,i_2,\cdots i_n}cA_{1i_1}cA_{2i_2}\cdots cA_{ni_n} = c^n\epsilon_{i_1,i_2,\cdots i_n}A_{1i_1}A_{2i_2}\cdots A_{ni_n} \\ =& c^n\det A\end{aligned} \tag{0.6}$$

5. For a general $k\times k$ matrix M show that

   a) $\det e^M = e^{trM}$.

   b) Use this to conclude that for $g = e^M$ we have $\log\det g = tr\log g$. Here the log of a matrix is defined as the inverse function of the exponential.

   Solution:

   (a) We can start as in the lecture to find

   $$\det e^M = \lim_{n\to\infty}\left[\det(\mathbb{1}+M/n)\right]^n \tag{0.7}$$

   Now consider $\det(\mathbb{1}+M/n)$. The terms appearing in the determinant are all products of entries of the matrix $\mathbb{1}+M/n$. Terms that do not contain diagonal elements have a factor n^{-k}, whereas each diagonal factor in a summand of the determinant replaces one factor of n^{-1} by one factor $(1+M_{ii}/n)$ (where i is the index of the diagonal element). The leading terms in the limit $n\to\infty$ are hence coming from the term in the determinant which has only factors from the diagonal of $\mathbb{1} + M/n$:

$$\lim_{n\to\infty} [\det(\mathbb{1} + M/n)]^n = \lim_{n\to\infty} \left[\prod_i \left(1 + \frac{M_{ii}}{n}\right)\right]^n \tag{0.8}$$

A similar feature appears when expanding the product, the leading terms are those that contain only a single factor of n^{-1}:

$$\prod_i \left(1 + \frac{M_{ii}}{n}\right) = 1 + \frac{1}{n}\sum_i M_{ii} + \mathcal{O}(n^{-2}) \tag{0.9}$$

Hence

$$\det e^M = \lim_{n\to\infty} [\det(\mathbb{1} + M/n)]^n = \lim_{n\to\infty} \left[1 + \frac{trM}{n}\right]^n = e^{trM} \tag{0.10}$$

(b) We have $g = e^M$ and $M = \log g$. Applying the above formula we find

$$\log\det g = \log\det e^M = \log e^{trM} = trM = tr\log g\,. \tag{0.11}$$

In the physics literature, the relationship proven in a) is mostly stated in the above form as 'log det = tr log'.

6. Show that the Pauli matrices

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{0.12}$$

satisfy

$$[\sigma_i, \sigma_j] = 2i\epsilon_{ijk}\sigma_k\,. \tag{0.13}$$

Solution:

We work out

$$\begin{aligned}[\sigma_1, \sigma_2] = \sigma_1\sigma_2 - \sigma_2\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} - \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} = 2i\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = 2i\sigma_3 = 2i\epsilon_{123}\sigma_3\end{aligned} \tag{0.14}$$

$$\begin{aligned}[\sigma_2, \sigma_3] &= \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix} - \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} = 2i\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = 2i\sigma_1 = 2i\epsilon_{231}\sigma_1\end{aligned} \tag{0.15}$$

$$
\begin{aligned}
[\sigma_3, \sigma_1] &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} - \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 2i \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} = 2i\sigma_2 = 2i\epsilon_{312}\sigma_2
\end{aligned}
\tag{0.16}
$$

The remaining cases of commutators follow from the antisymmetry of both the commutator and the ϵ_{ijk}. This is particular implies that both sides of the equation vanish whenever two indices are identical.

Here are some things to ponder:

1. What is the group $SU(2)$, what might $SU(n)$ be like?