1. Let f be a homomorphism between two groups G and H. Show that

   a) $f(e_G) = e_H$ where e_G and e_H are the unit elements of G and H, respectively.

   b) $f(g^{-1}) = f(g)^{-1}$ for any $g \in G$.

   solution:

   (a) Consider $f(g \circ g') = f(g) \circ f(g')$ and set $g' = e_G$. We then have that $f(g) = f(g) \circ f(e_G)$. Note that this looks like the definition of the identity, but only for those $h \in H$ in the image of f. To make sure, we can now multiply this equation with the inverse of $f(g)$, $f(g)^{-1}$ (which exists because H is a group) to find

   $$f(g)^{-1} \circ f(g) = f(g)^{-1} \circ f(g) \circ f(e_G) \quad \rightarrow \quad e_H = f(e_G) \tag{0.1}$$

   where we have used associativity.

   (b) Consider $f(g \circ g') = f(g) \circ f(g')$ and set $g' = g^{-1}$. We then find

   $$f(g \circ g') = f(e_G) = f(g) \circ f(g^{-1}) \,. \tag{0.2}$$

   As we have seen that $f(e_G) = e_H$ it must be that $f(g^{-1}) = f(g)^{-1}$.

2. $U(2)$ is the group of complex 2×2 matrices g such that $g^\dagger = g^{-1}$, with the group composition being matrix multiplication. Let F be the map which sends

   $$g \mapsto \det g \,. \tag{0.3}$$

   Show that F is a group homomorphism from $U(2)$ to $U(1)$.
   solution:
   First we need to verify that this maps indeed to $U(1)$. We can work out

   $$1 = \det \mathbb{1} = \det g^\dagger g = \det g^\dagger \ \det g = \overline{\det g} \det g = |\det g|^2 \tag{0.4}$$

   so that for $g \in U(2)$ we have that $|\det g|^2 = 1$ so that $\det g$ is in $U(1)$.

   As both groups are multiplicative, we need to check that $F(gh) = F(g)F(h)$. This follows from

   $$F(gh) = \det gh = \det g \ \det h = F(g)F(h) \,. \tag{0.5}$$

3. (a) Show that

   $$\sigma_i \sigma_j + \sigma_j \sigma_i = 2\delta_{ij}\mathbb{1} \tag{0.6}$$

   where σ_i are the Pauli matrices.

(b) Show that

$$g(\boldsymbol{\alpha}) = e^{i\boldsymbol{\alpha\sigma}} = \begin{pmatrix} \cos(a) + i\sin(a)a_3/a & \sin(a)a_2/a + i\sin(a)a_1/a \\ -\sin(a)a_2/a + i\sin(a)a_1/a & \cos(a) - i\sin(a)a_3/a \end{pmatrix} \tag{0.7}$$

where $a = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$. [hint: write $\boldsymbol{\alpha} = a\boldsymbol{n}$ with $|\boldsymbol{n}|^2 = 1$, i.e. $n_j = \alpha_j/a$]

solution:

(a) It is obvious that $\sigma_i^2 = \mathbb{1}$, so we only need to check the above eq. for $i \neq j$.

$$\begin{aligned} \{\sigma_1, \sigma_2\} = \sigma_1\sigma_2 + \sigma_2\sigma_1 &= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} + \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = 0 \end{aligned} \tag{0.8}$$

and similarly for all the remaining cases.

(b) Expanding the exponential we get

$$g(\boldsymbol{\alpha}) = e^{i\alpha_j\sigma_j} = \sum_{k=0}^{\infty} \frac{(i\alpha_j\sigma_j)^k}{k!} = \mathbb{1} + i\alpha_j\sigma_j + \tfrac{1}{2}(i\alpha_j\sigma_j)^2 + \cdots \tag{0.9}$$

It seems we cannot hope to get a nice formula as before due to the cross terms, e.g. $\alpha_1\alpha_2(\sigma_1\sigma_2 + \sigma_2\sigma_1)$. Happily, we know that

$$\sigma_i\sigma_j + \sigma_j\sigma_i = 2\delta_{ij}\mathbb{1} \tag{0.10}$$

so that the cross terms all cancel!

Let us now write $\boldsymbol{\alpha} = a\boldsymbol{n}$ with $|\boldsymbol{n}|^2 = 1$, i.e. $n_j = \alpha_j/a$ and $a = \sqrt{\alpha_1^2 + \alpha_2^2 + \alpha_3^2}$. We can then write

$$(\boldsymbol{n} \cdot \boldsymbol{\sigma})^2 = (n_1\sigma_1 + n_2\sigma_2 + n_3\sigma_3)^2 = (n_1^2 + n_2^2 + n_3^2)\,\mathbb{1} = \mathbb{1}\,. \tag{0.11}$$

Hence we find

$$\begin{aligned} g(\boldsymbol{\alpha}) = \sum_{k=0}^{\infty} \frac{(i\alpha_j\sigma_j)^k}{k!} &= \sum_{k=\text{even}}^{\infty} \frac{(ia\boldsymbol{n}\cdot\boldsymbol{\sigma})^k}{k!} + \sum_{k=\text{odd}}^{\infty} \frac{(ia\boldsymbol{n}\cdot\boldsymbol{\sigma})^k}{k!} \\ &= \sum_{k=\text{even}}^{\infty} \frac{(ia)^k}{k!}\mathbb{1} + \sum_{k=\text{odd}}^{\infty} \frac{(ia)}{k!}\boldsymbol{n}\cdot\boldsymbol{\sigma} = \cos(a)\mathbb{1} + i\sin(a)\boldsymbol{n}\cdot\boldsymbol{\sigma} \end{aligned} \tag{0.12}$$

When $g \neq \mathbb{1}$ can write this out as

$$g(\boldsymbol{\alpha}) = e^{i\boldsymbol{\alpha}\boldsymbol{\sigma}} = \begin{pmatrix} \cos(a) + i\sin(a)a_3/a & \sin(a)a_2/a + i\sin(a)a_1/a \\ -\sin(a)a_2/a + i\sin(a)a_1/a & \cos(a) - i\sin(a)a_3/a \end{pmatrix} \tag{0.13}$$

Note that these have the general form of matrices in $SU(2)$ derived in the lectures, as expected.

Here are some things to ponder:

1. What is the point about group homomorphisms and group isomorphisms?
2. What is a one-parameter subgroup?
3. What nice properties do the Pauli matrices have?