10) Consider the following action

$$S = \int dt \, \mathrm{Tr} \left( \dot{q}^2 - \omega q^2 \right)$$

for $q \in \mathfrak{su}(2)$ and $\omega \in \mathbb{R}$, i.e. we can write $q(t) = \sum_a q_a(t)\sigma_a$ with $\sigma_a$ the Pauli matrices and $q_a(t)$ real.

(a) Find the equations of motion by writing down the Euler-Lagrange equations which follow from $S$.

(b) Show that this action is invariant under $SU(n)$ acting as

$$q \to UqU^\dagger \qquad \dot{q} \to U\dot{q}U^\dagger$$

for $U \in SU(n)$.

(c) Find the conserved quantities under the $SU(n)$ action.

11) Consider the following action of a real scalar field $\phi(x^\mu)$

$$S = \int d^4x \, \partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \, .$$

Show that the equations of motion are

$$(-\partial_\mu \partial^\mu + m^2)\phi = 0 \, .$$

12) Consider the action

$$S = \int d^4x \bar{\Psi} \left( \gamma^\mu \partial_\mu + m \right) \Psi \, .$$

for a Dirac spinor field $\Psi(x^\mu)$.

(a) Find the equations of motion. [ **hint: $\Psi(x^\mu)$ has four complex components $\Psi_I$. Treat the $\Psi_I$ and $\bar{\Psi}_J$ as eight independent fields.**]

(b) The equations of motions have the form $D(m)\Psi = 0$. Show that $D(m)D(-m) = \mathbb{1}_{4\times4}\,\Delta$ for a $\Delta$ that you should find.

Here are some things to ponder:

1. What is an action?

2. What is a symmetry of an action?