10) Consider the following action

$$S = \int dt\, \mathrm{Tr}\left(\dot{q}^2 - \omega q^2\right) \tag{0.1}$$

for $q \in \mathfrak{su}(2)$ and $\omega \in \mathbb{R}$, i.e. we can write $q(t) = \sum_a q_a(t)\sigma_a$ with σ_a the Pauli matrices and $q_a(t)$ real.

(a) Find the equations of motion by writing down the Euler-Lagrange equations which follow from S.

(b) Show that this action is invariant under $SU(n)$ acting as

$$q \to UqU^\dagger \qquad \dot{q} \to U\dot{q}U^\dagger \tag{0.2}$$

for $U \in SU(n)$.

(c) Find the conserved quantities under the $SU(n)$ action.

solution:

(a) First notice that Tr $\sigma_a\sigma_b = 2\delta_{ab}$. We can hence write (using summation convention)

$$S = 2\int dt\, \dot{q}_a\dot{q}_a - \omega q_a q_a \tag{0.3}$$

and the equations of motion for each of the q_a are

$$\ddot{q}_a - \omega q_a = 0 \tag{0.4}$$

i.e. we can equivalently write

$$\ddot{q} - \omega q = 0\,. \tag{0.5}$$

(b) We have

$$\mathrm{Tr} q^2 \to \mathrm{Tr} UqU^\dagger UqU^\dagger = \mathrm{Tr} Uq^2U^\dagger = \mathrm{Tr} q^2 \tag{0.6}$$

using the cyclical property of the trace. Similarly

$$\mathrm{Tr}\dot{q}^2 \to \mathrm{Tr} U\dot{q}U^\dagger U\dot{q}U^\dagger = \mathrm{Tr} U\dot{q}^2U^\dagger = \mathrm{Tr}\dot{q}^2 \tag{0.7}$$

so L and hence S are invariant.

(c) We need to work out the formula for the Noether charge:

$$Q(\gamma) = \frac{\partial L}{\partial \dot{q}_i}\left(\rho(\gamma)\boldsymbol{q}\right)_i - F(q,\dot{q},\gamma) \tag{0.8}$$

note that part (b) shows that $F = 0$. Let us try to find the associated Lie algebra representation of the adjoint action. We have that a path in $SU(2)$ acts as

$$q \to e^{i\sigma_b\alpha_b t} q e^{-i\sigma_b\alpha_b t} \tag{0.9}$$

the associated Lie algebra action of which (= infinitesimal transformation) is

$$q \to \frac{\partial}{\partial t} e^{i\sigma_b\alpha_b t} q e^{-i\sigma_b\alpha_b t}\Big|_{t=0} \tag{0.10}$$

i.e.

$$q_a\sigma_a \to q_a + i[\sigma_b, \sigma_a]\alpha_b q_a = -2\epsilon_{abc}\alpha_b q_a \sigma_c = \left(\rho(i\sigma_b\alpha_b)q\right)_c \sigma_c \,. \tag{0.11}$$

As

$$\frac{\partial}{\partial \dot{q}_c} L = 2\dot{q}_c \tag{0.12}$$

we find the Noether charge

$$Q = -4\epsilon_{abc}\alpha_b q_a \dot{q}_c \tag{0.13}$$

Note that letting $\alpha = \alpha_a\sigma_a$ we can rewrite this more elegantly as follows:

$$Q = \mathrm{Tr}\left([g, \alpha]\dot{q}\right) = -2 q_a \alpha_b \dot{q}_d \epsilon_{abc} \mathrm{Tr}(\sigma_c\sigma_d) = -4\alpha_a q_b \dot{q}_c \epsilon_{abc} \tag{0.14}$$

Note that we find a conserved charge for any Lie algebra element, i.e. for any α. I total there are hence 3 independent conserved quantities. Note further that writing things in terms of the q_a things look like we just talking about $SO(3)$ again, so we get the same conserved things: collecting the q_a in a column vector $\boldsymbol{q}$ we can write the conserved charges as a linear combination of $\boldsymbol{q} \times \dot{\boldsymbol{q}}$, so this is like angular momentum in disguise :)

11) Consider the following action of a real scalar field $\phi(x^\mu)$

$$S = \int d^4x \, \partial_\mu\phi\partial^\mu\phi + m^2\phi^2 \,.$$

Show that the equations of motion are

$$(-\partial_\mu\partial^\mu + m^2)\phi = 0 \,.$$

solution:

We have

$$\partial\mathcal{L}/\partial\phi = 2m^2\phi \tag{0.15}$$

and

$$\begin{aligned}\frac{\partial}{\partial(\partial_\mu\phi)}\mathcal{L} &= \frac{\partial}{\partial(\partial_\mu\phi)}\ \partial_\rho\phi\partial^\rho\phi = \eta^{\rho\sigma}\frac{\partial}{\partial(\partial_\mu\phi)}\partial_\rho\phi\partial_\sigma\phi \\ &= \eta^{\rho\sigma}\partial_\sigma\phi\frac{\partial}{\partial(\partial_\mu\phi)}\ \partial_\rho\phi + \eta^{\rho\sigma}\partial_\rho\phi\frac{\partial}{\partial(\partial_\mu\phi)}\ \partial_\sigma\phi \\ &= \eta^{\rho\sigma}\delta^\mu{}_\rho\partial_\sigma\phi + \eta^{\rho\sigma}\delta^\mu{}_\sigma\partial_\rho\phi = 2\partial^\mu\phi\end{aligned} \tag{0.16}$$

The equation of motion for ϕ is hence

$$(-\partial_\mu\partial^\mu + m^2)\phi = 0 \tag{0.17}$$

Note that we can write the Lagrangian density as

$$\mathcal{L} = -(\frac{\partial}{\partial t}\phi)^2 + (\boldsymbol{\nabla}\phi)^2 + m^2\phi^2 \tag{0.18}$$

so this is really the same as example 4.2. You can check that the equations of motion are also the same in both cases.

12) Consider the action

$$S = \int d^4x\bar{\Psi}\left(\gamma^\mu\partial_\mu + m\right)\Psi\,.$$

for a Dirac spinor field $\Psi(x^\mu)$.

(a) Find the equations of motion. [**hint: $\Psi(x^\mu)$ has four complex components Ψ_I. Treat the Ψ_I and $\bar{\Psi}_J$ as eight independent fields.**]

(b) The equations of motions have the form $D(m)\Psi = 0$. Show that $D(m)D(-m) = \mathbb{1}_{4\times 4}\,\Delta$ for a Δ that you should find.

solution:

(a) To find the field equation for $\bar{\Psi}$, let us write out the Lagrangian in terms of the components Ψ_I of the spinors:

$$\mathcal{L} = \Psi_I^*\gamma^0_{IJ}\left(\gamma^\mu_{JK}\partial_\mu + \delta_{JK}m\right)\Psi_K \tag{0.19}$$

where γ^0_{IJ} and γ^μ_{JK} are the components of these matrices. The Euler-Lagrange equation for Ψ^* is simply

$$\frac{\partial\mathcal{L}}{\partial\Psi_I^*} = 0 \tag{0.20}$$

as there are no derivatives w.r.t Ψ^* in $\mathcal{L}$. We hence find

$$\gamma^0_{IJ}\left(\gamma^\mu_{JK}\partial_\mu + \delta_{JK} m\right)\Psi_K = 0\,. \tag{0.21}$$

Multiplying by $(\gamma^0)^{-1}$ gives

$$\left(\gamma^\mu\partial_\mu + m\right)\Psi = 0\,. \tag{0.22}$$

This is the famous Dirac equation.

(b) We work out

$$\begin{aligned}
(\gamma^\mu\partial_\mu - m)(\gamma^\nu\partial_\nu + m) &= \left(\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu - m^2\right)\\
&= \left(\tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + \tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu - m^2\right)\\
&= \left(\tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + \tfrac{1}{2}\partial_\nu\partial_\mu\gamma^\mu\gamma^\nu - m^2\right)\\
&= \left(\tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\mu\gamma^\nu + \tfrac{1}{2}\partial_\mu\partial_\nu\gamma^\nu\gamma^\mu - m^2\right)\\
&= \left(\tfrac{1}{2}\partial_\mu\partial_\nu\{\gamma^\mu,\gamma^\nu\} - m^2\right) = \left(\partial_\mu\partial_\nu\eta^{\mu\nu} - m^2\right)\\
&= \left(\partial_\mu\partial^\mu - m^2\right)
\end{aligned} \tag{0.23}$$

Note that we have simply relabelled μ and ν for the second term in the 4th line. The same result can be found by writing out the sums $\gamma^\mu\partial_\mu$ and $\gamma^\nu\partial_\nu$ and collecting all the terms. It is in the sense of the above equation that the Dirac equation is the square root of the Klein-Gordon equation. The above computation is what prompted Dirac to invent the Dirac matrices.

Here are some things to ponder:

1. What is an action?

2. What is a symmetry of an action?