1. Let G be the set of complex 2×2 matrices of the form

$$g = \begin{pmatrix} \alpha & \beta \\ -\bar{\beta} & \bar{\alpha} \end{pmatrix}$$

for $\alpha, \beta \in \mathbb{C}$ and $|\alpha|^2 + |\beta|^2 \neq 0$.

a) Show that G is a group using matrix multiplication as the group operation.

b) Show that $SU(2)$ is a subgroup of G.

c) Show that $V := \{\gamma | g = e^{i\gamma} \in G\}$ is a vector space (when ignoring the multi-valuedness of the logarithm) and find a basis for V.

solution:

a) We can do this in a straight-forward way by checking the group properties in this explicit form. A little more elegant is to realize that these are exactly the complex 2×2 matrices that obey

$$g^{\dagger} = g^{-1} \ \det(g) \tag{0.1}$$

with $\det g \neq 0$. Writing

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \tag{0.2}$$

this implies that

$$g^{\dagger} = \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} = \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \tag{0.3}$$

which results in the general form above.

Now this is clearly obeyed by the identity, if g obeys it then

$$(g^{-1})^{\dagger} = (g^{\dagger})^{\dagger} \det g = g \det g^{-1} \tag{0.4}$$

so the inverse is in G as well. Matrix multiplication is associative and finally

$$(gh)^{\dagger} = h^{\dagger} g^{\dagger} = h^{-1} \det h \ \ g^{-1} \det g = (gh)^{-1} \det gh \tag{0.5}$$

so that composition of group elements makes new group elements.

Remark: this is nothing but the group of quaternions written as complex matrices.

b) $SU(2)$ are those $g \in G$ with $\det g = |\alpha|^2 + |\beta|^2 = 1$. This feature is preserved when taking the inverse or multiplying two elements of $SU(2)$, so that $SU(2)$ is a subgroup.

c) There are two ways of approaching this. Let me first use part c), which immediately tells me that I can use $i\sigma_j$ with σ_j the Pauli matrices in the exponential. We can write any $g \in G$ that is also in $SU(2)$ as

$$g_{SU(2)} = \exp \sum_j i a_j \sigma_j \, . \tag{0.6}$$

Now this is all of it for $SU(2)$ which is real 3-dimensional (there are three real a_j), but how about the present case? Any element of G is determined by fixing the complex numbers α and β s.t. $|\alpha|^2 + |\beta|^2 \neq 0$ and this is four real parameters. We are hence looking for one more direction. What do matrices in G look like that are not in $SU(2)$? Here is an example: for any $\alpha = e^r$ with $r \neq 0$ we are not in $SU(2)$:

$$g = \begin{pmatrix} e^r & 0 \\ 0 & e^r \end{pmatrix} = \exp r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \, . \tag{0.7}$$

Now we can try writing a $g \in G$ as

$$g = \exp\left(\sum_j i a_j \sigma_j\right) \exp\left(r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = \exp\left(\sum_j i a_j \sigma_j + r \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) \tag{0.8}$$

as the identity matrix commutes with everything. We hence arrive at the set of all γs as

$$\left\{ \sum_j a_j \sigma_j - ir \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \middle| \, a_j \in \mathbb{R}, r \in \mathbb{R} \right\} \tag{0.9}$$

We are free to choose the a_j and r in the real numbers so that the set of all γ just described is $\mathbb{R}^4$, which is a vector space. Equally, you can show that addition and scalar multiplication preserves this set. A basis is given by $\sigma_j, j = 1, 2, 3$ and i times the identity matrix.

What is slightly unsatisfactory about this is that we don't know if we might have missed something, i.e. if the above is really V. What the above argument shows is that the general g we have constructed is a product of something in $SU(2)$ with the identity matrix times a positive number, so we can reach

$$e^{i\gamma} = g_{SU(2)} \begin{pmatrix} e^r & 0 \\ 0 & e^r \end{pmatrix} = e^r g_{SU(2)} \tag{0.10}$$

We can simply rescale any element in G by a positive number to reach an element in $SU(2)$, so the above is in fact general and we are done.

A faster way is to realize that

$$g^\dagger = (e^{i\gamma})^\dagger = e^{-i\gamma^\dagger} = g^{-1}\det g = e^{-i\gamma} e^{i\,\mathrm{tr}\gamma} \tag{0.11}$$

implies

$$\gamma^\dagger = \gamma - \mathbb{1}\mathrm{tr}\gamma \tag{0.12}$$

which forms a vector space: we have

$$(c\gamma)^\dagger = c\gamma - \mathbb{1}\mathrm{tr}c\gamma \tag{0.13}$$

for $c \in \mathbb{R}$ and

$$(\delta+\gamma)^\dagger = \delta^\dagger + \gamma^\dagger = \delta + \gamma - \mathbb{1}\,(\mathrm{tr}\gamma + \mathrm{tr}\delta) = \delta + \gamma - \mathbb{1}\mathrm{tr}\,(\gamma+\delta) \tag{0.14}$$

A basis of the vector space of solutions to $\gamma^\dagger = \gamma - \mathbb{1}\mathrm{tr}\gamma$ are the matrices we have found above, $\sigma_j, j = 1, 2, 3$ and $i\mathbb{1}$.

2. Which of the following sets are closed in the standard topology of $\mathbb{R}^m$? Which are open?

   (a) $\{0 < x < \pi\} \subset \mathbb{R}$ with coordinate x
   (b) $\{x_1 < -2\} \subset \mathbb{R}^2$ with coordinates (x_1, x_2)
   (c) $\{0 < x \leq \pi\} \subset \mathbb{R}$
   (d) $\{0 < x_1 < 1\} \subset \mathbb{R}^2$ with coordinates (x_1, x_2)
   (e) $\mathbb{R}^n \subseteq \mathbb{R}^n$
   (f) $\{(x_1, x_2) \subset \mathbb{R}^2 \,|\, x_1^2 \leq 42 - x_2^2\} \subset \mathbb{R}^2$ with coordinates (x_1, x_2)
   (g) $\{(x_1, x_2, x_3) | x_1^2 + x_2^2 = 1\} \subset \mathbb{R}^3$

   solution:

   (a) open, not closed
   (b) open, not closed
   (c) not open and not closed
   (d) open, not closed
   (e) open **and** closed; the same is true for the complement, $\emptyset$ is also both open and closed; these are the only such sets
   (f) this is a closed disc of radius $\sqrt{42}$, closed, not open
   (g) closed, not open

3. Prove that arbitrary unions and finite intersections of open sets in $\mathbb{R}^n$ are again open. Why is the intersection of an infinite number of open sets not open in general ?

   solution:

   Let $U = \bigcup_{u \in S} u$ be the union of an infinite set S of open sets u. Let $\boldsymbol{p}$ be any point in U. Then it must be contained in one of the u and hence there is an open ball entirely contained in u because u is open. As U is the union of all of these, this ball is also contained in U.

   For the second statement, let us start by considering a non-empty intersection between two open sets $U = U_1 \cap U_2$. For any point $\boldsymbol{p}$ in this intersection we can find a ball $B_{r_1}(\boldsymbol{p})$ centered at $\boldsymbol{p}$ that sits entirely in U_1, and a ball $B_{r_2}(\boldsymbol{p})$ that is entirely in U_2. Without loss of generality we can assume that $r_1 \leq r_2$, But this means that $B_{r_1}(\boldsymbol{p}) \subseteq B_{r_2}(\boldsymbol{p})$ so that $B_{r_1}(\boldsymbol{p}) \subset U$.

   Now let $U = \bigcap_{u \in S} u$ for a finite set S. Consider any point $\boldsymbol{p} \in U$. By repeating the above argument a finite number of times, we will find a finite sized open ball sitting in U.

   The latter argument fails for an arbitrary intersection $U_i, i \in \mathcal{N}$. Here, it can happen that the sizes r_i approach zero as $i \to \infty$. Letting $r_i \to 0$ for $i \to \infty$, the infinite intersection of open sets

   $$\bigcap_{i=1}^{\infty} B_{r_i}(\boldsymbol{p}) = \boldsymbol{p}$$

   is just a point, which is not an open set. Note that each r_i is finite, so each U_i is open.

Here are some things to ponder:

1. What is the relationship between $SO(3)$ and $SU(2)$?
2. How do open and closed subsets of $\mathbb{R}^m$ work?
3. What properties would you ask of open closed sets without reference to $\mathbb{R}^m$?