13) Assume that under

$$\begin{aligned} \phi_I &\to \phi_I + \delta_\gamma \phi_I = \left[(\mathbb{1} + \rho(\gamma))\, \boldsymbol{\phi} \right]_I \\ \partial_\mu \phi_I &\to \partial_\mu \phi_I + \delta_\gamma \partial_\mu \phi_I = \left[(\mathbb{1} + \rho(\gamma))\, \partial_\mu \boldsymbol{\phi} \right]_I \end{aligned} \tag{0.1}$$

$\mathcal{L}$ is not invariant but to linear order in $\delta\phi_I$ and $\delta\partial_\mu\phi_I$ we have

$$\mathcal{L} \to \mathcal{L} + \partial_\mu F^\mu(\phi_I, \partial_\nu \phi_I) \tag{0.2}$$

for some functions $F^\mu(\phi_I, \partial_\nu\phi_I)$.

By following the same steps as done in the proof of theorem 4.4., show that this also leads to a conserved current which you should find.

solution:

We have that

$$\begin{aligned} \partial_\mu F^\mu(\phi_I, \partial_\nu \phi_I) = \delta \mathcal{L} &= \frac{\partial \mathcal{L}}{\partial \phi_I} \delta_\gamma \phi_I + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_I} \partial_\mu \delta_\gamma \phi_I \\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_I} \right) \delta_\gamma \phi_I + \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_I} \partial_\mu \delta_\gamma \phi_I \\ &= \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_I} \delta_\gamma \phi_I \right) \end{aligned} \tag{0.3}$$

Hence

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_I} \delta_\gamma \phi_I - F^\mu \right) = 0 \tag{0.4}$$

which says using $\delta_\gamma \phi_I = [\rho(\gamma)\phi]_I$ that the current

$$j^\mu = \frac{\partial \mathcal{L}}{\partial \partial_\mu \phi_I} \left[\rho(\gamma) \phi \right]_I - F^\mu \tag{0.5}$$

is conserved.

14) Consider the action

$$S = \int d^4x \bar{\Psi} \left(\gamma^\mu \partial_\mu + m \right) \Psi \, .$$

for a Dirac spinor Ψ.

a) Show that S is Lorentz invariant.

b) Find the conserved charge associated to the $U(1)$ symmetry $\Psi \to e^{i\theta}\Psi$.

solution:

a) We have already seen the transformation behavior of all of the terms in this action when we replace ∂_μ by a constant Lorentz covector a_μ in the third problem class, where we found that they are all invariant. Transforming the argument of the spinor field Ψ effectively makes ∂_μ transform as a Lorentz covector as well, so that the above action is Lorentz invariant.

Let's translate the above into equations. We let $x^\mu \to \Lambda^\mu{}_\nu x^\nu$ and $\boldsymbol{y} = \Lambda^{-1}\boldsymbol{x}$, so that a Lorentz transformation maps

$$\Psi(\boldsymbol{x}) \to \Lambda_{1/2}\Psi(\boldsymbol{y})\,. \tag{0.6}$$

Note that $\Lambda_{1/2}$ is the 'spinor representation' matrix associated to Λ, i.e. if $\Lambda = e^{l^{\rho\sigma}\theta_{\rho\sigma}}$ then $\Lambda_{1/2} = e^{S^{\rho\sigma}\theta_{\rho\sigma}}$.

Using the transformation of $\bar{\Psi}$ studied before, $\bar{\Psi} \to \bar{\Psi}\Lambda_{1/2}^{-1}$ we find

$$\begin{aligned} S \to S' &= \int d^4x \bar{\Psi}(\boldsymbol{y})\Lambda_{1/2}^{-1}\left(\gamma^\mu \frac{\partial}{\partial x^\mu} + m\right)\Lambda_{1/2}\Psi(\boldsymbol{y}) \\ &= \int d^4y \bar{\Psi}(\boldsymbol{y})\Lambda_{1/2}^{-1}\left(\gamma^\mu \left(\Lambda^{-1}\right)^\rho{}_\mu \frac{\partial}{\partial y^\rho} + m\right)\Lambda_{1/2}\Psi(\boldsymbol{y}) \end{aligned}$$

where we have used the fact that the dervative behaves like a covector (via the product rule) and that $d^4x = d^4y$ for proper Lorentz transformations. Now we use the magical formula $\Lambda_{1/2}^{-1}\gamma^\mu\Lambda_{1/2} = \Lambda^\mu{}_\nu\gamma^\nu$. We then have

$$S' = \int d^4y \bar{\Psi}(\boldsymbol{y})\left(\gamma^\nu \left(\Lambda^{-1}\right)^\rho{}_\mu \Lambda^\mu{}_\nu \frac{\partial}{\partial y^\rho} + \Lambda_{1/2}^{-1}\Lambda_{1/2} m\right)\Psi(\boldsymbol{y}) \tag{0.7}$$

I have rearranged some factors (which are just numbers as we are using indices) and you can see that $(\Lambda^{-1})^\rho{}_\mu\Lambda^\mu{}_\nu = \delta^\rho{}_\nu$. As also $\Lambda_{1/2}^{-1}\Lambda_{1/2} = \mathbb{1}$ we end up with

$$S' = \int d^4y \bar{\Psi}(\boldsymbol{y})\left(\gamma^\nu \frac{\partial}{\partial y^\nu} + m\right)\Psi(\boldsymbol{y}) = S\,. \tag{0.8}$$

as it is now evident that all that has happened is that $\boldsymbol{x}$ has been relabelled as $\boldsymbol{y}$ everywhere.

b) Under the $U(1)$ symmetry acting on Ψ as $\Psi \to e^{i\theta}\Psi$, or in components $\Psi_I \to e^{i\theta}\Psi_I$. Ψ has 4 components Ψ_I, each of which is complex, so we need to treat Ψ_I and $\bar{\Psi}_I$ as 8 independent fields. The infinitesimal transformation are found by expanding to linear order in θ:

$$\delta\Psi_I = i\theta\Psi_I \qquad \delta\Psi_I = -i\theta\Psi_I \tag{0.9}$$

and the conserved current is (note we are using summation convention below, i.e. summing over I)

$$\begin{aligned} j^\mu &= \delta\Psi_I \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi_I)} + \delta\bar{\Psi}_I \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi}_I)} \\ &= i\theta\Psi_I \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Psi_I)} - i\theta\bar{\Psi}_I \frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\Psi}_I)} \\ &= i\theta\Psi_I\Psi^*_K\gamma^0_{KJ}\gamma^\mu_{JI} = i\theta\bar{\Psi}\gamma^\mu\Psi\,. \end{aligned} \tag{0.10}$$

Again this is conserved for any θ, and we don't loose anything rescaling the current to get rid of the $i\theta$ in the factor.
Rescaling this we get the conserved charge density $j^0 = -\bar{\Psi}\gamma^0\gamma^0\Psi = \Psi^*\Psi$, i.e. the conserved charge is

$$Q_V = \int_V d^3x\,|\Psi|^2 \tag{0.11}$$

which is positive definite. Hence one can use Ψ as a wave-function just as one does for the Schroedinger equation.

15) Consider a field Φ transforming in the adjoint representation of the Lie group $SU(n)$. Show that

$$S = \int d^4x \;\mathrm{tr}\,(\partial_\mu\Phi\partial^\mu\Phi)$$

is invariant under the action of $SU(n)$ and find the associated conserved current.

solution:

We first need to think about what it means to transform in the adjoint representation. The adjoint representation acts on the vector space that is equal to the Lie algebra of $SU(n)$. We should hence think of Φ as a (space-time dependent) element of the Lie algebra of $SU(n)$. In particular, this means Φ is a traceless anti-hermitian $n \times n$ matrix that transforms as

$$\Phi \to g\Phi g^{-1} \tag{0.12}$$

for $g \in SU(n)$ and also

$$\partial_\mu\Phi \to g(\partial_\mu\Phi)g^{-1} \tag{0.13}$$

Under this map

$$\begin{aligned}\mathrm{tr}\left(\partial_\mu\Phi\partial^\mu\Phi\right) &\to \mathrm{tr}\left(g\partial_\mu\Phi g^{-1}g\partial^\mu\Phi g^{-1}\right)\\ &= \mathrm{tr}\left(g\partial_\mu\Phi\partial^\mu\Phi g^{-1}\right) = \mathrm{tr}\left(g^{-1}g\partial_\mu\Phi\partial^\mu\Phi\right) \quad = \mathrm{tr}\left(\partial_\mu\Phi\partial^\mu\Phi\right)\end{aligned} \tag{0.14}$$

using the properties of the trace. The associated infinitesimal transformation (Lie algebra representation) is

$$\delta_\gamma\Phi = [\gamma,\Phi] \tag{0.15}$$

for $\gamma \in \mathfrak{su}(n)$. For a basis γ_i of the Lie algebra we can write

$$\Phi = \Phi_i\gamma_i \tag{0.16}$$

so that

$$\mathcal{L} = \partial_\mu\Phi_i\partial^\mu\Phi_j\mathrm{tr}\left(\gamma_i\gamma_j\right) \tag{0.17}$$

and

$$\gamma_i\delta_\gamma\Phi_i = \Phi_j[\gamma,\gamma_j] \tag{0.18}$$

We can now work out

$$\begin{aligned}j^\mu &= \frac{\partial\mathcal{L}}{\partial(\partial_\mu\Phi_i)}\delta_\gamma\phi_i = 2(\partial^\mu\Phi_j)\mathrm{tr}(\gamma_i\gamma_j)\delta_\gamma\Phi_i = 2\mathrm{tr}\left(\delta_\gamma\Phi\partial^\mu\Phi\right)\\ &= 2\mathrm{tr}\left([\gamma,\Phi]\partial^\mu\Phi\right)\end{aligned} \tag{0.19}$$

Another way to treat this is to uses the fact that we can think of $(\alpha,\beta) := -\mathrm{tr}\alpha\beta$ as an inner form (scalar product) on the vector space that is the Lie algebra. This implies we can choose an orthonormal basis γ_i which satisfies

$$\mathrm{tr}\gamma_i\gamma_j = -\delta_{ij}\,. \tag{0.20}$$

A third way to approach this is to redo the derivation of the equations of motion and Noethers theorem for fields which are matrices Φ starting from S.

All of these give the same answer of course.

Here are some things to ponder:

1. What does Noether's theorem tell you for a field theory?
2. When do we consider a physical system to be Lorentz invariant?