1. Consider the sets of points in $\mathbb{R}^2$ with coordinates (x, y) defined implicitely by the following relations

   a) $y = x^3$

   b) $xy = c$

   c) $x^2 + y^4 = 1$

   d) $x > y$

   e) $y^2 + x^3 - 3x - c = 0$

   Using the induced topology from $\mathbb{R}^2$, decide in each case if this is a differentiable manifold.
   [hint: plot them! Note that the word 'differentiable' here refers to the manifold and not the functions I used to define a manifold. The two notions are not unrelated however, details are explained in the non-examinable example 1.10. but this is not needed to answer this question.] **solution:**

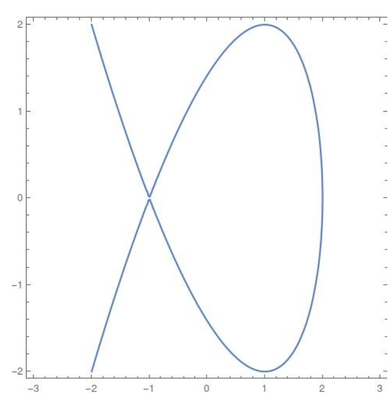
   a) This can be mapped to $\mathbb{R}$ using simply x as the coordinate, so this is in fact homeomorphic to $\mathbb{R}$ and it a manifold.

   b) For $c = 0$, this is the union of two lines $x = 0$ and $y = 0$ meeting at the origin and is not a manifold as discussed in the lectures. For all other values of c it is a manifold.

   c) This just looks like a dented circle and is a manifold.

   d) This has dimension two, but is a manifold; we can just use the coordinates of $\mathbb{R}^2$ used in its description.

   e) Let me call this set E. Plotting E for $c = 2$ reveals it looks like this



   E is an example of what is commonly called an 'elliptic curve'. As can be seen from the plot, two branches cross in the point $(y, x) = (0, -1)$.

This can be seen from the structure of the equation as well. For every x there are two values of y, except when

$$y^2 = -(x^3 - 3x - 2) = (2 - x)(1 + x)^2 = 0\,. \tag{0.1}$$

Note that double root at $x + 1 = 0$. We can write the above as

$$y = \pm(1 + x)\sqrt{2 - x}\,. \tag{0.2}$$

so that there are two branches which meet at $x = -1$. Zooming in on this point, it looks the same as the example of $xy = 0$ considered above, so that this cannot be a manifold for the same reasons. Something similar happens for $c = -2$. For all other c, the curve does not have this behavior and we in fact find a manifold.

2. Describe the tangent space of $SO(3)$ at the identity.

solution:

We can approach this in two ways. First, let me construct some paths in $SO(3)$ and then use these to find an expression for tangent vectors. Here is an element of $SO(3)$:

$$\begin{pmatrix} \cos\phi & \sin\phi & 0 \\ -\sin\phi & \cos\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.3}$$

As this is in $SO(3)$ for any ϕ, we can make this into a path which I will call Γ_3 (as this is a rotation around the x_3 axis) by simply relabelling ϕ into t which is in some interval $t \in (-1, 1)$:

$$\Gamma_3 : t \mapsto \begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.4}$$

It is important that this contains $t = 0$, as this is where we reach the identity, $\Gamma_3(0) = \mathbb{1}$. We can now work out the associated tangent vector as

$$T_{\mathbb{1}}(\Gamma_3) = \frac{\partial}{\partial t}\begin{pmatrix} \cos t & \sin t & 0 \\ -\sin t & \cos t & 0 \\ 0 & 0 & 1 \end{pmatrix}\Bigg|_{t=0} = \begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{0.5}$$

Note that we could have also used the description of a patch of $SO(3)$ using coordinates to describe the tangent vector, but I chose here to use the form above instead. In the end, tangent vectors are geometrical objects, and we are free to explicitely describe them in different ways.

We can now repeat the same for rotations around the x_1 or x_2 axis. Denoting the associated paths by Γ_1 and Γ_2 we find

$$T_{\mathbb{1}}(\Gamma_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} \qquad T_{\mathbb{1}}(\Gamma_2) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} \tag{0.6}$$

Now these are supposed to form a vector space of dimension 3, which is the dimension of $SO(3)$ which equals the dimension of its tangent spaces. Indeed the three tangent vectors we have found span a vector space of dimension 3 using addition of matrices (the vector space of antisymmetric real 3×3 matrices), so that we can conclude that

$$T_{\mathbb{1}}SO(3) = \left\{\gamma | \gamma^T = -\gamma\right\} \tag{0.7}$$

We can recover the same result as follows: we have already seen that writing

$$g = e^{\gamma} \tag{0.8}$$

for $g \in SO(3)$ implies that $\gamma^T = -\gamma$, and any such anti-symmetric γ gives us something in $SO(3)$ upon exponentiating. Hence we can write a path

$$\Gamma_\gamma : t \mapsto e^{t\gamma} \ , \qquad t \in (-1, 1) \, . \tag{0.9}$$

for any such γ. Note that multiplying an anti-symmetric matrix by a real number gives another anti-symmetric matrix, so $e^{t\gamma} \in SO(3)$ for all real t. Each such path passes through the identity for $t = 0$, so we find

$$T_{\mathbb{1}}(\Gamma_\gamma) = \frac{\partial}{\partial t} \, e^{t\gamma}\Big|_{t=0} = \gamma \, . \tag{0.10}$$

Hence the set of all tangent vectors is found again to be

$$T_{\mathbb{1}}SO(3) = \left\{\gamma | \gamma^T = -\gamma\right\} \, . \tag{0.11}$$

3. $O(1,1)$ are the real 2×2 matrices O which leave the bilinear form $x_1^2 - x_2^2$ invariant when acting on $\boldsymbol{x} = (x_1, x_2)$ as

$$\boldsymbol{x} \to O\boldsymbol{x} \, .$$

a) Show that $O(1,1)$ is a group using matrix multiplication.

b) Find the general form of elements of $O(1,1)$.

c) Explain why $O(1,1)$ is a differentiable manifold and write down coordinate charts.

d) Find the tangent space of $O(1,1)$ at the identity element.

solution:
Acting with $O \in O(1,1)$ on $\mathbb{R}^2$ is supposed to leave the bilinear form $x_1^2 - x_2^2$ invariant. Imagine you have found two matrics O, O' with this property. We can act first with O

$$\boldsymbol{x} \to O\boldsymbol{x} \tag{0.12}$$

which leaves $x_1^2 - x_2^2$ invariant, after which we can then act with O' which again leaves $x_1^2 - x_2^2$ invariant. In summary, we have then acted with $O'O$, which makes using matrix multiplication as the group composition a good idea.

(a) Let us first find a condition that must be satisfied by matrices in $O(1,1)$. We need

$$x_1^2 - x_2^2 = \boldsymbol{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{x} \tag{0.13}$$

to stay invariant. This is mapped to

$$\boldsymbol{x}^T O^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} O\boldsymbol{x} \overset{!}{=} \boldsymbol{x}^T \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \boldsymbol{x} \tag{0.14}$$

so we find the condition

$$O^T L O = L \tag{0.15}$$

with $L = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$

Let us now check the group property using matrix multipliction as $\circ$. The condition above is solved by O the identity matrix, so we have the identity element. This equation also implies that $\det O \neq 0$, so an inverse exists. We can even write it down: $O^{-1} = LO^TL$ because $LO^TLO = L^2 = \mathbb{1}$. Now multiplying $O^TLO = L$ by O^{-1} from the right and $(O^T)^{-1}$ from the left shows

$$L = (O^T)^{-1} L O^{-1} = (O^{-1})^T L O^{-1} \,. \tag{0.16}$$

so the inverse satisfies the same equation. We have used that $(O^T)^{-1} = (O^{-1})^T$ which can be seen by

$$(O^T)^{-1} O^T = \mathbb{1} = (OO^{-1})^T = (O^{-1})^T O^T \,. \tag{0.17}$$

Finally we can observe that for $O'' = OO'$ with $O, O' \in O(1,1)$ we have

$$(O'')^T L O'' = (O')^T O^T L O O' = (O')^T L O' = L\,. \qquad (0.18)$$

so $O'' \in O(1,1)$ as well.

(b) Note that $\det O = \pm 1$ by taking the determinant of $O^T L O = L$ on both sides. Let us denote $\det O = \delta$ which can take the 2 values ± 1. We can parametrize O as

$$O = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \qquad O^{-1} = \pm \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \qquad (0.19)$$

and plug this into $O^T L = L O^{-1}$ to find

$$a = \delta d\ , \qquad b = \delta c\,. \qquad (0.20)$$

The determinant of O changes continuously along any path in O, which implies that there are hence (at least) two components, one for $\delta = 1$ and one for $\delta = -1$.

<u>$\det O = 1$</u> This implies that

$$O = \begin{pmatrix} a & b \\ b & a \end{pmatrix} \qquad (0.21)$$

with $a^2 - b^2 = 1$. We can parametrize solutions of this as

$$a = \pm \cosh\phi \qquad b = \sinh\phi\,. \qquad (0.22)$$

There are two inequivalent solutions as $\cosh(\phi) > 0$, as $-\sinh(\phi) = \sinh(-\phi)$ the choice of sign of b does not make a difference. Note that the two types of solution we have found here are disjoint, a is always positive for one and always negative for the other. Hence we have found two disconnected components $O_+^\uparrow$ ($\det O > 0, a > 0$) and $O_+^\downarrow$ ($\det O > 0, a < 0$).

<u>$\det O = -1$</u> This implies that

$$O = \begin{pmatrix} a & b \\ -b & -a \end{pmatrix} \qquad (0.23)$$

with $-a^2 + b^2 = -1$.

Repeating the same steps as above we find $a = \pm\cosh(\phi)$ and $b = \sinh(\phi)$ for this case. Hence we get two more components $O_-^\uparrow$ ($\det O <$

$0, a > 0$) and $O_-^\downarrow$ ($\det O < 0, a < 0$).

In summary we have hence found 4 disjoint components. Only $O_+^\uparrow$ is a subgroup as it is the only component that contains the identity element.

(c) Note that we can map any component of $O(1,1)$ to a copy of $\mathbb{R}$ by the parametrization by ϕ in a one-to-one fashion. Let us call these maps $\Phi_\pm^{\uparrow\downarrow}$. These give us good coordinate charts as these are clearly one-to-one continuous maps. Note that the a subset is open if the corresponding subset is open in $\mathbb{R}$.

(d) The component connected to $\mathbb{1}$ is $O_+^\uparrow$ and we can describe a path through $\mathbb{1}$ by

$$O(t) = \begin{pmatrix} \cosh(t) & \sinh(t) \\ \sinh(t) & \cosh(t) \end{pmatrix} \tag{0.24}$$

for $t = -e..e$ with some $e > 0$. We work out

$$\frac{\partial}{\partial t} O(t)|_{t=0} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{0.25}$$

so that the tangent space at $\mathbb{1}$ is the vector space of matrices

$$T_{\mathbb{1}} = \left\{ \begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix} |v \in \mathbb{R} \right\} . \tag{0.26}$$

Note that

$$\exp\left[\begin{pmatrix} 0 & v \\ v & 0 \end{pmatrix}\right] = \begin{pmatrix} \cosh(v) & \sinh(v) \\ \sinh(v) & \cosh(v) \end{pmatrix} \tag{0.27}$$

Here are some things to ponder:

1. Why are homeomorphisms and manifolds defined the way they are?
2. Tangent spaces are linear approximations.