1. Writing a vector $(v_1, v_2, v_3) \in \mathbb{R}^3$ as
$$M_{\boldsymbol{v}} = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} .$$
consider the action of $g \in SU(2)$ on $\mathbb{R}^3$ defined by
$$F(g) : M_{\boldsymbol{v}} \mapsto g M_{\boldsymbol{v}} g^\dagger .$$
Show that this is a representation, and that this representation is the adjoint representation of $SU(2)$.

2. The adjoint action representation defines a linear map $r(g)$ acting on $\mathfrak{g}$ and as such can be written as a matrix M acting on a column vector after choosing a basis for $\mathfrak{g}$. Make this explicit for the action of
$$g = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \in SU(2) . \tag{0.1}$$
in the adjoint representation. Is the adjoint representation faithful?

3. Let P be a homogeneous polynomial in two complex variables z_1 and z_2 of degree d, i.e. we can write
$$P(\boldsymbol{z}) = \sum_{k=0}^{d} \alpha_k z_1^k z_2^{d-k} \tag{0.2}$$
for complex numbers α_k.
There is a natural action of $SU(2)$ on $\boldsymbol{z} = (z_1, z_2)$, which is just
$$\boldsymbol{z} \mapsto g\boldsymbol{z} . \tag{0.3}$$
For a polynomial $P(\boldsymbol{z})$, we can then define an action by $SU(2)$ as
$$r_d(g) : P(\boldsymbol{z}) \mapsto P(g^{-1}\boldsymbol{z}) . \tag{0.4}$$
Show that this defines a representation of $SU(2)$.
[remark: in the above formula, g^{-1} does not act on the argument of P but on $\boldsymbol{z}$, i.e. the action on $P(A\boldsymbol{z})$ for a 2×2 matrix A would be
$r_d(g) : P(A\boldsymbol{z}) \mapsto P(Ag^{-1}\boldsymbol{z})$.]

Here are some things to ponder:

1. What are representations?

2. How can you define a representation?