1. Writing a vector $(v_1, v_2, v_3) \in \mathbb{R}^3$ as

$$M_{\boldsymbol{v}} = \begin{pmatrix} v_3 & v_1 - iv_2 \\ v_1 + iv_2 & -v_3 \end{pmatrix} .$$

consider the action of $g \in SU(2)$ on $\mathbb{R}^3$ defined by

$$F(g) : M_{\boldsymbol{v}} \mapsto g M_{\boldsymbol{v}} g^{\dagger} .$$

Show that this is a representation, and that this representation is the adjoint representation of $SU(2)$.

solution:

Clearly, the set of matrices above forms a vector space which is isomorphic to $\mathbb{R}^3$. Choosing the Pauli matrices as a basis, we can write

$$M_{\boldsymbol{v}} = \sum_j v_j \sigma_j \tag{0.1}$$

for $v_j \in \mathbb{R}$. Furthermore, we can describe this as the vector space of complex 2×2 matrices with $tr\, M_v = 0$ and $M_v^{\dagger} = M_v$. Both of these properties are preserved by $M_{\boldsymbol{v}} \mapsto g M_{\boldsymbol{v}} g^{\dagger}$:

$$\begin{array}{c} tr g M_{\boldsymbol{v}} g^{\dagger} = tr g^{\dagger} g M_{\boldsymbol{v}} = tr M_{\boldsymbol{v}} = 0 \\ (g M_{\boldsymbol{v}} g^{\dagger})^{\dagger} = (g^{\dagger})^{\dagger} M_{\boldsymbol{v}}^{\dagger} g^{\dagger} = g M_{\boldsymbol{v}} g^{\dagger} \end{array} . \tag{0.2}$$

Finally, the map $F(g)$ acts linearly on $M_{\boldsymbol{v}}$, so that $F : SU(2) \to GL(3, \mathbb{R})$. The only thing left to show to have a representation is that F is a homomorphism. We have

$$F(gh) : M_{\boldsymbol{v}} \mapsto gh M_{\boldsymbol{v}} (gh)^{\dagger} = gh M_{\boldsymbol{v}} h^{\dagger} g^{\dagger} \tag{0.3}$$

which is just the composition of the maps $F(h)$ and $F(g)$ acting on $M_{\boldsymbol{v}}$, so that this is a group homomorphism. More explicitely, if we write the action of $F(g)$ as a matrix acting $\boldsymbol{v}$, the above must be matrix multiplication, i.e. we can then write $F(gh) = F(g)F(h)$.

As defined in the lectures, the adjoint representation of G acts on $\mathfrak{g}$ as

$$\gamma \to g \gamma g^{-1} . \tag{0.4}$$

For $\mathfrak{su}(2)$, $g^{-1} = g^{\dagger}$ and we can write

$$\gamma = i \sum_j v_j \sigma_j . \tag{0.5}$$

The action of $F(g)$ on the v_j is hence the same as above (despite the extra factor of i) which means that $F(g)$ and the adjoint represention take g to the same elements of $GL(3,\mathbb{R})$, so that we conclude that F is the same as the adjoint representation.

2. The adjoint action representation defines a linear map $r(g)$ acting on $\mathfrak{g}$ and as such can be written as a matrix M acting on a column vector after choosing a basis for $\mathfrak{g}$. Make this explicit for the action of

$$g = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} \in SU(2)\,. \tag{0.6}$$

in the adjoint representation. Is the adjoint representation faithful?
solution: We have to work out the adjoint action on $\mathfrak{su}(2)$ in detail. We can write

$$\gamma = \sum_j i\alpha_j\sigma_j = i\begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_3 \end{pmatrix} \tag{0.7}$$

for three real numbers α_j. Note that using $i\sigma_j$ as a basis of $\mathfrak{su}(2)$, we could also represent γ as a column vector $(\alpha_1, \alpha_2, \alpha_3)$.

This is mapped to

$$i\begin{pmatrix} \alpha_3 & \alpha_1 - i\alpha_2 \\ \alpha_1 + i\alpha_2 & \alpha_3 \end{pmatrix} \to i\begin{pmatrix} \alpha_3 & e^{2i\phi}(\alpha_1 - i\alpha_2) \\ e^{-2i\phi}(\alpha_1 + i\alpha_2) & \alpha_3 \end{pmatrix}\,. \tag{0.8}$$

The action on the α_j is hence

$$\begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \to \begin{pmatrix} \cos 2\phi & \sin 2\phi & 0 \\ -\sin 2\phi & \cos 2\phi & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \equiv M \begin{pmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_3 \end{pmatrix} \tag{0.9}$$

Note that we did exactly the same computation already in section 1.1.4 in the lectures! There we also realized that both g and $-g$ are mapped to the same M, so the adjoint of $SU(2)$ is not injective. Note that the same applies to the adjoint of any group (if $-g \in G$ for $g \in G$):

$$(-g)\gamma(-g)^{-1} = (-1)^2 g\gamma g^{-1} = g\gamma g^{-1} \tag{0.10}$$

3. Let P be a homogeneous polynomial in two complex variables z_1 and z_2 of degree d, i.e. we can write

$$P(\boldsymbol{z}) = \sum_{k=0}^{d} \alpha_k z_1^k z_2^{d-k} \tag{0.11}$$

for complex numbers α_k.

There is a natural action of $SU(2)$ on $\boldsymbol{z} = (z_1, z_2)$, which is just

$$\boldsymbol{z} \mapsto g\boldsymbol{z}\,. \tag{0.12}$$

For a polynomial $P(\boldsymbol{z})$, we can then define an action by $SU(2)$ as

$$r_d(g) : P(\boldsymbol{z}) \mapsto P(g^{-1}\boldsymbol{z})\,. \tag{0.13}$$

Show that this defines a representation of $SU(2)$.

[remark: in the above formula, g^{-1} does not act on the argument of P but on $\boldsymbol{z}$, i.e. the action on $P(A\boldsymbol{z})$ for a 2×2 matrix A would be
$r_d(g) : P(A\boldsymbol{z}) \mapsto P(Ag^{-1}\boldsymbol{z})$.]

solution:

We need to check four things: i) we are acting on a vector space Π_d ii) that this map indeed maps elements of Π_d to Π_d, iii) that it is linear, iv) that it is a group homomorphism from $SU(2)$ to $GL(\Pi_d)$.

i) we can write

$$P(\boldsymbol{z}) = \sum_{k=0}^{d} a_k z_1^k z_2^{d-k}\,.$$

for any such polynomial. Adding two of these or multiplying by a complex number just adds or rescales the a_k, so this defines a vector space which we can call Π_d. You can think of the a_k as the components of the vectors and the monomials as basis vectors. As there are $d+1$ different monomials for a polynomial of degree d, this is a complex vector space of dimension $d+1$.

ii) Here, it is enough to observe that g^{-1} acts linearly on $\boldsymbol{z}$. Hence it preserved the degree of P so that it indeed maps any element of Π_d to another element of Π_d.

iii) Note that

$$r_d(g)(P+Q) = (P+Q)(g^{-1}\boldsymbol{z}) = P(g^{-1}\boldsymbol{z}) + Q(g^{-1}\boldsymbol{z}) = r_d(g)(P) + r_d(g)(Q) \tag{0.14}$$

Hence $r_{d+1}(g)$ acts linearly on Π_d.

iv) Consider the action of $r_d(g)r_d(h)$ on P:

$$r_d(g)r_d(h)P(z) = r_d(g)P(h^{-1}z) = P(h^{-1}g^{-1}\boldsymbol{z}) = P((gh)^{-1}\boldsymbol{z}) = r_d(gh)P\,. \tag{0.15}$$

We hence have a group homomorphism[1]. To see that it is in $GL(\Pi_d)$ you might be concerned about $r_d(g)P = 0$ for some g. That this does not happen follows from the fact that we are talking about a group homomorphism: if $r_d(g)P = 0$ then also $r_d(g^{-1})r_d(g)P = 0$. But $r_d(g^{-1})r_d(g)P = r_d(g^{-1}g)P = P$, which is a contradiction.

As a final comment, note the peculiar $g^{-1}\boldsymbol{z}$ instead of $g\boldsymbol{z}$. The deeper reason for this is that we are acting on the basis vectors (monomials) instead of components as usual.

Here are some things to ponder:

1. What are representations?
2. How can you define a representation?

[1] Note that it is for this reason we needed the $g^{-1}\boldsymbol{z}$.