

- 23) A magnetic monopole of magnetic charge  $m$  located at the origin  $O$  of three-dimensional space is described by a divergence-free magnetic field  $\vec{B}$  in  $\mathbb{R}^3 \setminus O$ , with non-vanishing magnetic flux through the 2-sphere that surrounds the origin  $O$ :

$$\frac{1}{2\pi} \int_{S^2} \mathbf{B} \cdot d\boldsymbol{\sigma} = m \neq 0.$$

- (a) Show that all of the above can be reformulated as the equation

$$\nabla \cdot \mathbf{B} = 2\pi m \delta^{(3)}(\vec{x})$$

in  $\mathbb{R}^3$ .

- (b) Using that

$$\nabla \frac{1}{r} = -\frac{\mathbf{x}}{r^3}, \quad \Delta \frac{1}{r} = -4\pi \delta^{(3)}(\mathbf{x}),$$

where  $r = |\mathbf{x}|$  and  $\Delta = \nabla^2$  is the Laplacian, show that

$$\mathbf{B} = \frac{m}{2} \frac{\mathbf{x}}{r^3}$$

solves the equation in part (a).

- (c) For

$$A_x^\pm = \mp \frac{m}{2} \frac{y}{r(r \pm z)}, \quad A_y^\pm = \pm \frac{m}{2} \frac{x}{r(r \pm z)}, \quad A_z^\pm = 0. \quad (1)$$

Show that the corresponding magnetic field is

$$\nabla \times \mathbf{A}^\pm = \frac{m}{2} \frac{\mathbf{x}}{r^3}. \quad (2)$$

- 24) Write the  $A^\pm = A_x^\pm dx + A_y^\pm dy + A_z^\pm dz$  in (1) in spherical coordinates as

$$A^\pm = \frac{m}{2} (\pm 1 - \cos \theta) d\varphi$$

using  $dx = \frac{\partial x}{\partial r} dr + \frac{\partial x}{\partial \theta} d\theta + \frac{\partial x}{\partial \varphi} d\varphi$  and similarly for  $dy$ .

- 25) The energy stored in electromagnetic fields is

$$\frac{1}{2} \int d^3x (\mathbf{E}^2 + \mathbf{B}^2).$$

Show that the energy of the magnetic monopole solution (2) is infinite. How about an electric monopole?

Here are some things to ponder:

1. What are key aspects of abelian gauge theories?
2. How can we get away with defining a magnetic monopole in a  $U(1)$  gauge theory?