23) A magnetic monopole of magnetic charge m located at the origin O of three-dimensional space is described by a divergence-free magnetic field $\boldsymbol{B}$ in $\mathbb{R}^3 \backslash O$, with non-vanishing magnetic flux though the 2-sphere that surrounds the origin O:

$$\frac{1}{2\pi}\int_{S^2} \boldsymbol{B}\cdot d\boldsymbol{\sigma} = m \neq 0.$$

(a) Show that all of the above can be reformulated as the equation

$$\nabla\cdot\boldsymbol{B} = 2\pi m\ \delta^{(3)}(\boldsymbol{x})$$

in $\mathbb{R}^3$.

(b) Using that

$$\nabla\frac{1}{r} = -\frac{\boldsymbol{x}}{r^3}\ , \qquad \Delta\frac{1}{r} = -4\pi\ \delta^{(3)}(\boldsymbol{x})\ ,$$

where $r = |\boldsymbol{x}|$ and $\Delta = \nabla^2$ is the Laplacian, show that

$$\boldsymbol{B} = \frac{m}{2}\frac{\boldsymbol{x}}{r^3}$$

solves the equation in part (a).

(c) For

$$A_x^{\pm} = \mp\frac{m}{2}\frac{y}{r(r\pm z)}\ , \quad A_y^{\pm} = \pm\frac{m}{2}\frac{x}{r(r\pm z)}\ , \quad A_z^{\pm} = 0\ . \tag{1}$$

Show that the corresponding magnetic field is

$$\nabla\times\boldsymbol{A}^{\pm} = \frac{m}{2}\frac{\boldsymbol{x}}{r^3}\ . \tag{2}$$

solution:

(a) If $\boldsymbol{x}\neq\boldsymbol{0}$ then $\nabla\cdot\boldsymbol{B} = 0$ is equivalent to $\nabla\cdot\boldsymbol{B} = 2\pi m\ \delta^{(3)}(\boldsymbol{x})$.
If we extend $\mathbb{R}^3\backslash O$ to $\mathbb{R}^3$, a 2-sphere S^2 (of any radius) surrounding the origin is the boundary of a 3-ball B^3 of the same radius, and Gauss' theorem gives

$$\int_{S^2}\boldsymbol{B}\cdot d\boldsymbol{\sigma} = \int_{B^3}\nabla\cdot\boldsymbol{B}\ d^3x\ .$$

The volume integral on the right-hand side can only receive contribution from the origin, because $\boldsymbol{B}$ is divergence-free elsewhere. (Alternatively, the flux of the magnetic field through a 2-sphere surrounding the origin is independent of the radius of the sphere.) Then it must be that

$$\nabla \cdot \boldsymbol{B} = c\ \delta^{(3)}(\boldsymbol{x})$$

for some constant c.[1] Plugging this in the above equation and comparing with the given magnetic flux we find

$$2\pi m = \int_{S^2} \boldsymbol{B} \cdot d\boldsymbol{\sigma} = \int_{B^3} \nabla \cdot \boldsymbol{B}\ d^3x = c \int_{B^3} \delta^{(3)}(\boldsymbol{x})\ d^3x = c\ .$$

(b) We just need to remember that the Laplacian is the divergence of the gradient, or mathematically $\Delta = \nabla \cdot \nabla$. Then

$$\begin{aligned} \nabla \cdot \left(\frac{m}{2} \frac{\boldsymbol{x}}{r^3} \right) &= -\frac{m}{2} \nabla \cdot \left(-\frac{\boldsymbol{x}}{r^3} \right) = -\frac{m}{2} \nabla \cdot \nabla \frac{1}{r} = -\frac{m}{2} \Delta \frac{1}{r} \\ &= -\frac{m}{2} (-4\pi)\ \delta^{(3)}(\boldsymbol{x}) = 2\pi m\ \delta^{(3)}(\boldsymbol{x})\ . \end{aligned}$$

(c) We can do the computation in two ways here: using Cartesian coordinates, with $r = \sqrt{x^2 + y^2 + z^2}$; or using polar coordinates, which make the gauge field easier to write and its field strength easier to calculate, but then we need to switch back to Cartesian coordinates.

Let's use Cartesian coordinates, which is the approach I expect most of you will have taken. We start by writing

$$A_x^{\pm} = -\pm \frac{m}{2} \frac{y}{r(r + \pm z)}\ , \qquad A_y^{\pm} = \pm \frac{m}{2} \frac{x}{r(r + \pm z)}\ , \qquad A_z^{\pm} = 0$$

with $\pm^2 = 1$, so that we can treat the two cases $\pm = \pm 1$ in one go. We'll also use

$$\partial_x r = \frac{x}{r}\ , \qquad \partial_y r = \frac{y}{r}\ , \qquad \partial_z r = \frac{z}{r}\ ,$$

and write r for $\sqrt{x^2 + y^2 + z^2}$.

[1] To be precise, derivatives of the delta function would also be allowed, but it's easy to see that they wouldn't reproduce the desired magnetic flux. I would give full marks even if this is not noticed.

Next we calculate the derivatives

$$\begin{aligned}
\partial_x A_y^\pm &= \pm\frac{m}{2}\left[\frac{1}{r(r+\pm z)} - x\frac{x/r}{r^2(r+\pm z)} - x\frac{x/r}{r(r+\pm z)^2}\right]\\
&= \pm\frac{m}{2}\frac{1}{r^3(r+\pm z)^2}\left[r^2(r+\pm z) - x^2(r+\pm z) - x^2 r\right]\\
\partial_y A_x^\pm &= -\pm\frac{m}{2}\frac{1}{r^3(r+\pm z)^2}\left[r^2(r+\pm z) - y^2(r+\pm z) - y^2 r\right]\\
\partial_z A_y^\pm &= \pm\frac{m}{2}x\left[-\frac{z/r}{r^2(r+\pm z)} - \frac{z/r}{r(r+\pm z)^2} - \frac{\pm}{r(r+\pm z)^2}\right]\\
&= -\pm\frac{m}{2}\frac{x}{r^3(r+\pm z)^2}\left[z(r+\pm z) + zr + \pm r^2\right]\\
&= -\pm\frac{m}{2}\frac{x}{r^3(r+\pm z)^2}\pm(r+\pm z)^2 = -\frac{m}{2}\frac{x}{r^3}\\
\partial_z A_x^\pm &= \frac{m}{2}\frac{y}{r^3}\\
\partial_y A_z^\pm &= \partial_x A_z = 0\ .
\end{aligned}$$

From this we find the components of $\nabla\times\boldsymbol{A}^\pm$:

$$\begin{aligned}
(\nabla\times\boldsymbol{A}^\pm)_x &= \partial_y A_z^\pm - \partial_z A_y^\pm = \frac{m}{2}\frac{x}{r^3}\\
(\nabla\times\boldsymbol{A}^\pm)_y &= \partial_z A_x^\pm - \partial_x A_z^\pm = \frac{m}{2}\frac{y}{r^3}\\
(\nabla\times\boldsymbol{A}^\pm)_z &= \partial_x A_y^\pm - \partial_y A_x^\pm\\
&= \pm\frac{m}{2}\frac{1}{r^3(r+\pm z)^2}\left[(r+\pm z)(2r^2 - r^2 + z^2) - r(r^2 - z^2)\right]\\
&= \pm\frac{m}{2}\frac{1}{r^3(r+\pm z)}\left[r^2 + z^2 - r(r-\pm z)\right]\\
&= \pm\frac{m}{2}\frac{1}{r^3(r+\pm z)}\pm z(r+\pm z) = \frac{m}{2}\frac{z}{r^3}
\end{aligned}$$

as required.

24) Write the $A^\pm = A_x^\pm dx + A_y^\pm dy + A_z^\pm dz$ in (1) in spherical coordinates as

$$\boldsymbol{A} = \frac{m}{2}(\pm 1 - \cos\theta)d\varphi$$

using $dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial\theta}d\theta + \frac{\partial x}{\partial\varphi}d\varphi$ and similarly for dy. **solution:**

Spherical coordinates are defined by

$$\begin{aligned} x &= r\sin\theta\cos\varphi \\ y &= r\sin\theta\sin\varphi \\ z &= r\cos\theta \end{aligned} \tag{1.1}$$

Using $dx = \frac{\partial x}{\partial r}dr + \frac{\partial x}{\partial \theta}d\theta + \frac{\partial x}{\partial \varphi}d\varphi$ and similarly for dy we find

$$\begin{aligned} &\frac{2}{m}\left(A^\pm_x dx + A^\pm_y dy + A^\pm_z dz\right) \\ &= dr\left(-\pm\frac{r\sin\theta\sin\varphi}{r(r\pm z)}\sin\theta\cos\varphi \pm \frac{r\sin\theta\cos\varphi}{r(r\pm z)}\sin\theta\sin\varphi\right) \\ &+ d\theta\left(-\pm\frac{r^2\sin\theta\sin\varphi}{r(r\pm z)}\cos\theta\cos\varphi \pm \frac{r^2\sin\theta\cos\varphi}{r(r\pm z)}\cos\theta\sin\varphi\right) \\ &+ d\varphi\left(-\pm\frac{r\sin\theta\sin\varphi}{r(r\pm z)}(-r\sin\theta\sin\varphi) \pm \frac{r\sin\theta\cos\varphi}{r(r\pm z)}(r\sin\theta\cos\varphi)\right) \\ &= \frac{1}{1\pm\cos\theta}\left(\pm\sin^2\theta\sin^2\varphi \pm \sin^2\theta\cos^2\varphi\right)d\varphi \\ &= \frac{\pm(1-\cos^2\theta)}{1\pm\cos\theta}d\varphi = (\pm 1 - \cos\theta)d\varphi \end{aligned} \tag{1.2}$$

25) The energy stored in electromagnetic fields is

$$\mathcal{E} = \tfrac{1}{2}\int d^3x(\boldsymbol{E}^2 + \boldsymbol{B}^2)\,.$$

Show that the energy of the magnetic monopole solution (2) is infinite. How about an electric monopole?

solution:
Using spherical coordinates we have

$$\frac{1}{2}\int dV\boldsymbol{B}^2 = \frac{1}{2}\int r^2 dr d\varphi d\theta\sin\theta\frac{m^2}{4}\frac{r^2}{r^6} = \frac{\pi}{2}m^2\int dr\, r^{-2} = \frac{\pi}{2}m^2\left[-1/r\right]_0^\infty \tag{1.3}$$

which is not finite at $r = 0$. We can make the integral start at $r = \epsilon$ which makes it finite, showing that the point-like nature of the monopole is the problem making the energy diverge. For an electric monopole, one would write

$$\boldsymbol{E} = \frac{q}{2}\frac{\boldsymbol{x}}{r^3}$$

for some charge q. As the energy is symmetrical w.r.t. electric and magnetic fields, the answer will be same as above, i.e. electric point charges have infinite energy stored in their fields.

Here are some things to ponder:

1. How can we get away with defining a magnetic monopole in a $U(1)$ gauge theory?
2. How would you go about defining a non-abelian gauge theory?