1. Let $\mathbf{q} \in \mathbb{C}^n$ be acted on in the fundamental representation of $SU(n)$ and γ in the adjoint representation of $SU(n)$ (this is often expressed as $\mathbf{q}$ 'lives' in the fundamental and γ 'lives' in the adjoint of $SU(n)$.)

   By acting with $SU(n)$ simultaneously on γ and $\boldsymbol{q}$, describe the action of $SU(n)$ on

   i) $\mathbf{v} = \gamma\mathbf{q}$

   ii) $\bar{\mathbf{q}}$

   iii) A matrix Q with components $Q_{ij} = q_i q_j$

   and decide in each case if this defines a representation.

   solution:

   i) $SU(n)$ acts on both γ and $\mathbf{q}$ and sends

   $$\mathbf{v} = \gamma\mathbf{q} \to g\gamma g^{-1} g\mathbf{q} = g\gamma\mathbf{q} = g\boldsymbol{v}\,. \tag{0.1}$$

   The set of all elements $\gamma\mathbf{q}$ forms a vector space which is just $\mathbb{C}^n$, so that this is again the defining representation of $SU(n)$.

   ii) Here we have

   $$\bar{\mathbf{q}} \to \bar{g}\bar{\mathbf{q}} \tag{0.2}$$

   and the set of all $\bar{\mathbf{q}}$ is again $\mathbb{C}^n$. This defines a map from $SU(n)$ to $GL(n,\mathbb{C})$ given by $r(g) = \bar{g}$. Let's check this is a representation by checking it is a homomorphism:

   $$r(gh) = \overline{gh} = \bar{g}\bar{h} = r(g)r(h)\,. \tag{0.3}$$

   iii) In this case the action is (note use of summation convention)

   $$Q_{ij} = q_i q_j \to g_{ik} q_k g_{jl} q_l \tag{0.4}$$

   which in matrix language is

   $$Q \to gQg^T\,. \tag{0.5}$$

   To see if this can give us a representation, let us first examine if matrices of the form of Q form a vector space. For this to be the case, we need that there is a $\boldsymbol{q}''$ such that for every $\boldsymbol{q}$ and $\boldsymbol{q}'$ we can write for all i,j

   $$q''_i q''_j = q_i q_j + q'_i q'_j \tag{0.6}$$

These are $n(n+1)/2$ independent relations in n complex variables, which have no solution for $n > 1$. (you can also note that setting $i = j$ already fixes all q_i'' and we cannot solve the remaining relations.). So matrices of the form Q do not form a vector space and we have nothing to talk about in terms of representations. However, they naturally sit inside the vector space V of all $n \times n$ matrices, or all symmetric $n \times n$ matrices if we want to be more restrictive, which form vector spaces. We can hence use this to define a representation by extending the scope and acting on all matrices $Q \in V$ as

$$Q \to gQg^T \,. \tag{0.7}$$

Note that gh then acts as

$$Q \to ghQ(gh)^T = ghQh^Tg^T \tag{0.8}$$

which is just the composition of the maps $r(h)$ and $r(g)$ on Q, so that we have a homomorphism from $SU(n)$ to $GL(V)$ and hence a representation.

2. Consider the map $r_\kappa : U(1) \to GL(3, \mathbb{C})$ defined by

$$r_\kappa(e^{i\phi}) = e^{\phi\lambda\kappa}$$

where $\kappa \in \mathbb{C}$ and

$$\lambda = \begin{pmatrix} 0 & i & 0 \\ i & 0 & i \\ 0 & i & 0 \end{pmatrix}$$

For which values of κ is r_κ a representation of $U(1)$? [hint: think about what happens to eigenvectors of λ and use the classification theorem for complex representations of $U(1)$.]

solution: First note that

$$r(e^{i\phi})r(e^{i\psi}) = e^{\phi\lambda\kappa}e^{\psi\lambda\kappa} = e^{(\phi+\psi)\lambda\kappa}$$

so this looks like a homomorphism. Parametrizing $U(1)$ as we did, we also need to make sure that $r(e^{2\pi i}) = \mathbb{1}$. This is not obvious immediately.

We know that if this is a representation, we can decompose it into irreducible representations, which are one-dimensional. We are hence looking for three invariant subspace of $\mathbb{C}^3$, which we can construct from eigenvectors of λ. Whenever

$$\lambda \boldsymbol{v} = c\boldsymbol{v}$$

we find that for any vector $a\boldsymbol{v}$ proportional to $\boldsymbol{v}$:

$$e^{\phi\lambda\kappa}a\boldsymbol{v} = e^{\phi c\kappa}a\boldsymbol{v}\,. \tag{0.9}$$

The eigenvalues of λ are $\pm i\sqrt{2}$ and 0, and the eigenvectors are $(1, \pm\sqrt{2}, 1)$ and $(-1, 0, 1)$. The subspace spanned by $(-1, 0, 1)$ carries a trivial representation, but the other two subspaces are acted on by

$$e^{\phi i\sqrt{2}\kappa}\,. \tag{0.10}$$

We hence need $\sqrt{2}\kappa = n$ with $n \in \mathbb{Z}$ for this to be a representation, which implies

$$\kappa = n/\sqrt{2}\,.$$

3. Let G be a Lie group and H be a subgroup of G that is also a Lie group.

   a) Explain why any representation $r(G)$ of G also gives us a representation $r(H)$ of H.

   b) Let's assume $r(G)$ is irreducible. Can you think of an example where the representation $r(H)$ is reducible? Can you think of an example where the representation $r(H)$ is irreducible?

solution:

(a) As we have a representation of G, there is a group homomorphism

$$r : G \to GL(V) \tag{0.11}$$

for some vector space V so that

$$r(gg') = r(g)r(g')\,. \tag{0.12}$$

for all $g, g' \in G$. As H is a subgroup, we can simply restrict ourselves to consider only element $h, h' \in H$. As H is a subgroup, $hh' \in H$ for all $h, h' \in H$. Hence

$$r(hh') = r(h)r(h') \tag{0.13}$$

for all $h, h' \in H$, so that we get a group homomorphism and hence a representation.

(b) To find an example where this becomes reducible, consider the group $SU(3)$ and its subgroup $SU(2)$ of matrices of the form

$$\begin{pmatrix} a & b & 0 \\ -\bar{b} & \bar{a} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{0.14}$$

with $|a|^2 + |b|^2 = 1$. Taking the defining representation of $SU(3)$, the $SU(2)$ subgroup acts on $\mathbb{C}^3$ as written above, which leaves the subspace of vectors of the form $(0, 0, z_3)$ invariant, so that it is reducible.

To find an example where the representation stays irreducible, consider the defining representation of $O(3)$. This is clearly irreducible. But $O(3)$ has a subgroup $SO(3)$, and restricting the defining representation of $O(3)$ to $SO(3)$ gives the defining representation of $SO(3)$ which is also irreducible.