1. a) Describe a $U(1)$ subgroup of $SU(2)$. Is $U(1) \times U(1)$ a subgroup of $SU(2)$ as well?

   b) Let A be an element of the vector space that is acted on by the adjoint representation of $SU(2)$. For the $U(1)$ subgroup of $SU(2)$ you identified above, find the action on A and use this to decompose the action of $U(1)$ into irreducible representations.

2. Consider the Lie group G of upper triangular 2×2 matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{R}, ac \neq 0 \right\} \tag{0.1}$$

   a) Let $\boldsymbol{v} \in \mathbb{R}^3$, $\boldsymbol{v} = (v_1, v_2, v_3)$. Define an action of G on $\boldsymbol{v}$ by writing

$$v_m := \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} \tag{0.2}$$

   and letting $g \in G$ act as

$$r(g)v_m := g v_m g^{-1} . \tag{0.3}$$

   Convince yourself that this is a representation of G. Write the action of $r(g)$ on $\boldsymbol{v}$ defined above in terms of a 3×3 matrix acting on $\boldsymbol{v}$:

$$r(g)\boldsymbol{v} = M(g)\boldsymbol{v} \tag{0.4}$$

   for a 3×3 matrix $M(g)$ acting on the vector $\boldsymbol{v} \in \mathbb{R}^3$ in the usual way.

   b) Writing $r(g)$ in terms of the matrices $M(g)$, work out the associated representation ρ of the Lie algebra $\mathfrak{g}$ of G.

   c) Check that they give a Lie algebra representation of the Lie algebra $\mathfrak{g}$ of G, i.e. find a Lie algebra homomorphism between the Lie algebra $\mathfrak{g}$ of G and the Lie algebra representation $\rho(\mathfrak{g})$ associated with $r(G)$.

3. Show that any irreducible complex representation of $SO(3)$ also defines an irreducible complex representation of $SU(2)$.

Here are some things to ponder:

1. What is the relationship between representations of Lie groups and Lie algebras?

2. What are all the complex irreducible representations of $SU(2)$? How might one proceed to construct complex irreducible representations of $SU(3)$ or other Lie groups?