26) Show that

$$i[D_\mu, D_\nu] = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] \tag{1.1}$$

solution:

Restoring the identity matrix $\mathbb{1}$ for clarity (feel free to omit it if you are comfortable without it),

$$\begin{aligned}[D_\mu, D_\nu] &= [\mathbb{1}\partial_\mu - iA_\mu, \mathbb{1}\partial_\nu - iA_\nu] \\ &= [\mathbb{1}\partial_\mu, \mathbb{1}\partial_\nu] - i[\mathbb{1}\partial_\mu, A_\nu] - i[A_\mu, \mathbb{1}\partial_\nu] - [A_\mu, A_\nu] \\ &= 0 - i(\partial_\mu A_\nu) + i(\partial_\nu A_\mu) - [A_\mu, A_\nu] \\ &= -i\left(\partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]\right) \ .\end{aligned}$$

27) By considering infinitesimal gauge transformations ($|\alpha^a| \ll 1$)

$$g = e^{i\alpha^a t_a} \equiv e^{i\alpha} = 1 + i\alpha + O(\alpha^2) \tag{1.2}$$

and Taylor expanding finite gauge transformations to leading order in $\alpha \in \mathfrak{g} = \mathrm{Lie}(G)$, show that the **infinitesimal gauge variations** of the fields are

$$\begin{aligned}\delta_\alpha \phi &= i\alpha\phi \\ \delta_\alpha A_\mu &= i[\alpha, A_\mu] + \partial_\mu \alpha \\ \delta_\alpha F_{\mu\nu} &= i[\alpha, F_{\mu\nu}] \ ,\end{aligned} \tag{1.3}$$

where $\phi \mapsto \phi + \delta_\alpha \phi$ and so on to leading order.

solution: By construction, these have to give us the associated Lie algebra representations (but now space-time dependent): Expanding to linear order

$$\delta\phi = e^{i\alpha}\phi - \phi \simeq (1 + i\alpha)\phi - \phi = i\alpha\phi \tag{1.4}$$

$$\begin{aligned}\delta A_\mu &= e^{i\alpha}(A_\mu + i\partial_\mu)e^{-i\alpha} - A_\mu \\ &\simeq (1 + i\alpha)(A_\mu + i\partial_\mu)(1 - i\alpha) - A_\mu \\ &\simeq i\alpha A_\mu - iA_\mu\alpha + \partial_\mu\alpha = i[\alpha, A_\mu] + \partial_\mu \alpha \, .\end{aligned} \tag{1.5}$$

where we have used that $\alpha\partial_\mu\alpha \ll \partial_\mu\alpha$. Finally

$$\begin{aligned}\delta F_{\mu\nu} &= e^{i\alpha}F_{\mu\nu}e^{-i\alpha} - F_{\mu\nu} \\ &\simeq (1 + i\alpha)F_{\mu\nu}(1 - i\alpha) - F_{\mu\nu} \\ &\simeq i\alpha F_{\mu\nu} - iF_{\mu\nu}\alpha = i[\alpha, F_{\mu\nu}] \, .\end{aligned} \tag{1.6}$$

29) Consider a gauge group G, with Lie algebra $\mathfrak{g}$.

(a) Show by explicit calculation that a non-abelian gauge field configuration of the form

$$A_\mu = ih(\partial_\mu h^{-1}) \ ,$$

where $h(x)$ is a space-time dependent element of G, has field strength $F_{\mu\nu} = 0$.

solution:

We calculate

$$\partial_\mu A_\nu = i(\partial_\mu h)(\partial_\nu h^{-1}) + ih(\partial_\mu \partial_\nu h^{-1}) \ ,$$

therefore

$$\begin{aligned} \partial_\mu A_\nu - \partial_\nu A_\mu &= i(\partial_\mu h)(\partial_\nu h^{-1}) - i(\partial_\nu h)(\partial_\mu h^{-1}) + ih(\partial_\mu \partial_\nu h^{-1}) - ih(\partial_\nu \partial_\mu h^{-1}) \\ &= i(\partial_\mu h)(\partial_\nu h^{-1}) - i(\partial_\nu h)(\partial_\mu h^{-1}) \ , \end{aligned}$$

where the second derivative terms cancel (as usual, we assume that h^{-1} is sufficiently differentiable so that Schwarz's/Clairaut's theorem applies). The contribution of the commutator is

$$\begin{aligned} -i[A_\mu, A_\nu] &= i[h\partial_\mu h^{-1}, h\partial_\nu h^{-1}] \\ &= ih(\partial_\mu h^{-1})h(\partial_\nu h^{-1}) - ih(\partial_\nu h^{-1})h(\partial_\mu h^{-1}) \ . \end{aligned}$$

Now we use the identity

$$0 = (\partial_\mu \mathbf{1}) = \partial_\mu (hh^{-1}) = (\partial_\mu h)h^{-1} + h(\partial_\mu h^{-1})$$

to get

$$\begin{aligned} -i[A_\mu, A_\nu] &= -i(\partial_\mu h)h^{-1}h(\partial_\nu h^{-1}) + i(\partial_\nu h)h^{-1}h(\partial_\mu h^{-1}) \\ &= -i(\partial_\mu h)(\partial_\nu h^{-1}) + i(\partial_\nu h)(\partial_\mu h^{-1}) \ . \end{aligned}$$

Hence

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu] = 0 \ .$$

(b) Can you think of a simpler argument to reach the same conclusion?

solution:

Start from a configuration with vanishing gauge field $A_\mu = 0$. The field strength trivially vanishes: $F_{\mu\nu} = 0$. Now perform a gauge transformation with gauge parameter $g = h$. We find that the new (gauge transformed) gauge field A'_μ and field strength $F'_{\mu\nu}$ are

$$\begin{aligned} A'_\mu &= hA_\mu h^{-1} + ih(\partial_\mu h^{-1}) = ih(\partial_\mu h^{-1}) \\ F'_{\mu\nu} &= hF_{\mu\nu}h^{-1} = 0 \ . \end{aligned}$$

Now, what is primed or unprimed is a matter of point of view: I could have called the primed variables unprimed and vice versa, had I used the inverse gauge transformation. The key point here is that this shows that the field strength of $A_\mu = ih(\partial_\mu h^{-1})$ is $F_{\mu\nu} = 0$. Configurations like $A_\mu = ih(\partial_\mu h^{-1})$, which are obtained by a gauge transformation of the trivial (*i.e.* zero) configuration, are called *pure gauge* configurations.

Here are some things to ponder:

1. How are convariant derivative and field strength defined for a non-abelian gauge theory?

2. What is the impact of charged matter in different representations?