1. a) Describe a $U(1)$ subgroup of $SU(2)$. Is $U(1) \times U(1)$ a subgroup of $SU(2)$ as well?

   b) Let A be an element of the vector space that is acted on by the adjoint representation of $SU(2)$. For the $U(1)$ subgroup of $SU(2)$ you identified above, find the action on A and use this to decompose the action of $U(1)$ into irreducible representations.

solution:

(a) We can simply take matrices of the form

$$g(\phi) = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} . \tag{0.1}$$

These are in $SU(2)$ for any ϕ and are isomorphic to $U(1)$. There is no subgroup $U(1) \times U(1)$ in $SU(2)$, and we argue as follows. For $U(1) \times U(1)$ there are two generators, one for each $U(1)$ factor, and these generators commute with each other. Otherwise we would be talking about a different group. We are hence looking for two $U(1)$ subgroups of $SU(2)$ which commute. We can write these as

$$e^{i\phi_1 \alpha} \text{ and } e^{i\phi_2 \beta} \tag{0.2}$$

for some real linear combinations of Pauli matrices $\alpha = \sum_i a_i \sigma_i$ and $\beta = b_i \sigma_i$. Now for these to commute we need

$$\left[e^{i\phi_1 \alpha}, e^{i\phi_2 \beta}\right] = 0 \tag{0.3}$$

for all values of ϕ_1 and ϕ_2. Taking derivatives with respect to ϕ_1 and ϕ_2 this implies

$$[\alpha, \beta] = 0 \tag{0.4}$$

which is clearly also a sufficient condition. We work out

$$[\alpha, \beta] = [a_i \sigma_i, b_j \sigma_j] = 2i\epsilon_{ijk} a_i b_j \sigma_k \tag{0.5}$$

which only vanishes when $\epsilon_{ijk} a_i b_j = 0$ for all k. The only solution except $\alpha = 0$ or $\beta = 0$ (which is inacceptable since then we don't get $U(1) \times U(1)$) is $a_i = b_i$ for all i. But then we generate the same $U(1)$ twice.

Here is another nice solution that some of you came up with: $U(1) \times U(1)$ has four elements that square to the identity:

$$(1, 1), (-1, 1), (1, -1), (-1, -1) .$$

If $U(1) \times U(1)$ were a subgroup of $SU(2)$, at least four such elements must exist in $SU(2)$ as well. By using the general form of $SU(2)$ you can see that $g \in SU(2)$ squaring to the identity implies $|a|^2 = 1$ and $b = 0$, so these matrices must be diagonal, and there only two such matrices which square to the identity.

(b) The vector space we act on in the adjoint is the vector space of matrices A such that $A^\dagger = -A$ and $trA = 0$. The action on this is

$$A \to gAg^\dagger = \begin{pmatrix} e^{i\phi} & 0 \\ 0 & e^{-i\phi} \end{pmatrix} A \begin{pmatrix} e^{-i\phi} & 0 \\ 0 & e^{i\phi} \end{pmatrix} . \tag{0.6}$$

for the choice of $U(1)$ made above. We find

$$\begin{aligned} A_{11} \to& A_{11} \\ A_{21} \to& e^{-2i\phi} A_{21} \\ A_{12} \to& e^{2i\phi} A_{12} \\ A_{22} \to& A_{22} \end{aligned} \tag{0.7}$$

Hence $A_{11} = -A_{22}$ is an invariant subspace of charge 0. $A_{12} = -\bar{A}_{21}$ is another invariant subspace of charge 2.

2. Consider the Lie group G of upper triangular 2×2 matrices

$$G = \left\{ \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} | a, b, c \in \mathbb{R}, ac \neq 0 \right\} \tag{0.8}$$

a) Let $\boldsymbol{v} \in \mathbb{R}^3$, $\boldsymbol{v} = (v_1, v_2, v_3)$. Define an action of G on $\boldsymbol{v}$ by writing

$$v_m := \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} \tag{0.9}$$

and letting $g \in G$ act as

$$r(g)v_m := g v_m g^{-1} . \tag{0.10}$$

Convince yourself that this is a representation of G. Write the action of $r(g)$ on $\boldsymbol{v}$ defined above in terms of a 3×3 matrix acting on $\boldsymbol{v}$:

$$r(g)\boldsymbol{v} = M(g)\boldsymbol{v} \tag{0.11}$$

for a 3×3 matrix $M(g)$ acting on the vector $\boldsymbol{v} \in \mathbb{R}^3$ in the usual way.

b) Writing $r(g)$ in terms of the matrices $M(g)$, work out the associated representation ρ of the Lie algebra $\mathfrak{g}$ of G.

c) Check that they give a Lie algebra representation of the Lie algebra $\mathfrak{g}$ of G, i.e. find a Lie algebra homomorphism between the Lie algebra $\mathfrak{g}$ of G and the Lie algebra representation $\rho(\mathfrak{g})$ associated with $r(G)$.

solution:

(a) First of all, this is a map that maps a matrix of the form v_m to another one of this form as product of upper triangular 2×2 matrices are again upper triangular as seen in problem 18. Furthermore it acts on the v_i linearly:

$$g(v_m + v_m')g^{-1} = g(v_m)g^{-1} + g(v_m')g^{-1} \tag{0.12}$$

so r defines a map from G to $GL(3,\mathbb{R})$. What is left to check is that r is a group homomorphism. We work out

$$r(gh)v_m = ghv_m(gh)^{-1} = ghv_m h^{-1}g^{-1} = r(g)r(h)v_m\,. \tag{0.13}$$

which shows it is. Hence this is a real three-dimensional representation of G. It is not injective as both g and $-g$ are mapped to $\mathbb{1} \in GL(3,\mathbb{R})$. Note that this is simply the adjoint representation.

Let us work out explicitely how $r(g)$ acts on v_m:

$$\begin{aligned} v_m = \begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix} \to v_m' = gv_mg^{-1} &= \frac{1}{ac}\begin{pmatrix} a & b \\ 0 & c \end{pmatrix}\begin{pmatrix} v_1 & v_2 \\ 0 & v_3 \end{pmatrix}\begin{pmatrix} c & -b \\ 0 & a \end{pmatrix} \\ &= \frac{1}{ac}\begin{pmatrix} acv_1 & -v_1ab + a^2v_2 + abv_3 \\ 0 & acv_3 \end{pmatrix} \end{aligned} \tag{0.14}$$

Hence

$$\begin{aligned} v_1 &\to v_1 \\ v_2 &\to -b/cv_1 + a/cv_2 + b/cv_3 \\ v_3 &\to v_3 \end{aligned} \tag{0.15}$$

We can write this as the action of a 3×3 matric on a column vector as

$$\boldsymbol{v} \to \boldsymbol{v}' = M(g)\boldsymbol{v} = \begin{pmatrix} 1 & 0 & 0 \\ -b/c & a/c & b/c \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \tag{0.16}$$

(b) Using the family of paths $a = e^{xt}$, $b = ty$, $c = e^{zt}$, $x, y, z \in \mathbb{R}$, we get

$$M\left(g(t)\right) = \begin{pmatrix} 1 & 0 & 0 \\ -tye^{-zt} & e^{t(x-z)} & tye^{-tz} \\ 0 & 0 & 1 \end{pmatrix} \tag{0.17}$$

and

$$\left.\frac{\partial}{\partial t} M\left(g(t)\right)\right|_{t=0} = \begin{pmatrix} 0 & 0 & 0 \\ -y & x-z & y \\ 0 & 0 & 0 \end{pmatrix} \tag{0.18}$$

This defines the associated Lie algebra representation ρ. Hence for every γ in $\mathfrak{g}$ we can write

$$\rho(\gamma) = x\rho(\ell_x) + y\rho(\ell_y) + z\rho(\ell_z) \tag{0.19}$$

where

$$\rho(\ell_x) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho(\ell_y) = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \quad \rho(\ell_z) = -\rho(\ell_x) \tag{0.20}$$

Note that the vector space they span is just two-dimensional and that using different paths here might give you a different basis of the Lie algebra.

The matrices above satisfy

$$[\rho(\ell_x), \rho(\ell_y)] = \rho(\ell_y) \tag{0.21}$$

(c) Using the paths above in problem 18 you find that a general element of the Lie algebra of G can be written as

$$x\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + y\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} + z\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \equiv x\ell_x + y\ell_y + z\ell_z \tag{0.22}$$

for $x, y, z \in \mathbb{R}$. The algebra of the ℓ is

$$[\ell_x, \ell_y] = \ell_y \quad [\ell_x, \ell_z] = 0 \quad [\ell_z, \ell_y] = -\ell_y\,. \tag{0.23}$$

The same relations are obeyed by $\rho(\ell_x)$, $\rho(\ell_y)$ and $\rho(\ell_z)$. It is totally fine that ℓ_x and ℓ_z are mapped to the same generator up to a sign, a homomorphism does not need to be an isomorphism and can have a non-trivial kernel.

3. Show that any irreducible complex representation of $SO(3)$ also defines an irreducible complex representation of $SU(2)$.

solution:

Let us assume that we are given an irredicible representation $r_{SO(3)}$ of $SO(3)$, i.e.

$$r_{SO(3)} : SO(3) \to GL(n, \mathbb{C}) \tag{0.24}$$

is a homomorphism and there is no complex sub-vectorspace W of $\mathbb{C}^n$ (except $\mathbb{C}^n$ and $\{0\}$) s.t.

$$r_{SO(3)}(g)w \in W \;\; \forall w \in W, \forall g \in SO(3)\,. \tag{0.25}$$

As shown in the lectures there is a close relationship between $SO(3)$ and $SU(2)$, i.e. there is a homomorphism $\pi : SU(2) \to SO(3)$. We can hence define the following composition

$$r_{SU(2)} := r_{SO(3)} \circ \pi \tag{0.26}$$

which takes any $h \in SU(2)$ to an element of $SO(3)$ and then to an element of $GL(n, \mathbb{C})$, so in effect we are taking any $h \in SU(2)$ to an element of $GL(n, \mathbb{C})$. As compositions of homomorphisms are again homomorphisms, this is a homomorphism as well and hence defines a representation of $SU(2)$.

Now let's investigate irredicibility. As we have seen π is surjective, i.e. we can write any $g \in SO(3)$ as $\pi(h)$ for some $h \in SU(2)$. As there is no complex sub-vectorspace W of $\mathbb{C}^n$ (except $\mathbb{C}^n$ and $\{0\}$) s.t.

$$r_{SO(3)}(g)w \in W \;\; \forall w \in W, \forall g \in SO(3)\,. \tag{0.27}$$

and we can write any such g as $g = \pi(h)$, it follows that there is no complex sub-vectorspace W of $\mathbb{C}^n$ (except $\mathbb{C}^n$ and $\{0\}$) s.t.

$$r_{SU(2)}(h)w \in W \;\; \forall w \in W, \forall h \in SU(2)\,. \tag{0.28}$$

So $r_{SU(2)}$ is irredicible as well.

Here are some things to ponder:

1. What is the relationship between representations of Lie groups and Lie algebras?
2. What are all the complex irreducible representations of $SU(2)$? How might one proceed to construct complex irreducible representations of $SU(3)$ or other Lie groups?