

QFT II Lecture Notes

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1 Orientation

These lecture notes were written for QFT II in the Particles, Strings and Cosmology MSc course at Durham University. They are a basic introduction to the path integral approach to quantum field theory. Most of the material follows the exposition in the well-known textbooks [1–4]. Please send all typos to andreas.braun@durham.ac.uk.

It's assumed the reader has some exposure to quantum field theory in canonical quantization, and also knows how to apply path integrals to quantum mechanics.

1.1 Motivation

I want to begin with some philosophical words about quantum field theory. *Why* do we study quantum field theory? What is it that makes us struggle through this confusing, technically difficult, at times seemingly¹ ill-defined sea of ideas?

Presumably this question has many answers, but perhaps I'll start making an extremely incomplete list of a few things that QFT is good for:

1. **Particle physics and the Standard Model:** this is the main focus of this course (both the QFT course and the larger MSc course of which it is a part). The Standard Model is a wildly successful physical theory, it is perhaps the most precise microscopic description of nature ever, and quantum field theory is really at its very core.

Most textbooks lean heavily on this point of view, but it gives you the idea that QFT is only important if you care about high-energy physics, and that you need to build a giant particle accelerator to appreciate it. This is not true.

2. **Metals and superconductors:** this is maybe not obvious, but both metals and superconductors – real-life low-energy objects that you can find in your home (or, well, nearby physics lab) – are described by quantum field theories; they are just not *relativistic*. The low energy fluctuations of electrons in metals are described by something called *Fermi liquid theory*, and superconductors are actually described by an analogue of the Higgs mechanism that you will study in the SM part of this course.
3. **Boiling water:** the phase diagram of water looks like **picture on board**. The critical point at 374 C is actually described by a three-dimensional quantum field theory called the 3d Ising model. I don't have time to explain why this is, but in general continuous phase transitions (...of any sort) are described by quantum field theories.
4. **Quantum gravity:** if you don't care about boiling water, perhaps you like quantum gravity. The so-called AdS/CFT correspondence tells us that quantum gravity in D dimensions is *precisely equivalent* to a quantum field theory in $D - 1$ dimensions (in its most well-studied incarnation it is dual to $SU(N)$ gauge theory). By the end of this course you will know what $SU(N)$ gauge theory is.

One may ask what all of these phenomena have in common that makes them describable by quantum field theory. In essence, if you ever have a situation where there are both **fluctuations** – thermal, quantum, etc. and **locality** – in space, in time, etc. – in some sense, it is likely that some sort of quantum field theory describes your system.

Having said all of that, in this course we will describe only *relativistic quantum field theory*; as always, extra symmetries (such as those associated with relativistic invariance) simplify our lives.

¹But only seemingly.

1.2 Conventions

In agreement with most (but not all) books on quantum field theory, I will use the “mostly-minus” metric:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(1, -1, -1, -1) \quad (1.1)$$

Sadly this is the opposite convention to the one that I’m familiar with; thus everyone will need to be vigilant for sign errors. We will mostly work in four spacetime dimensions (the “physical” value), but it is nice to keep the spacetime dimension d arbitrary where possible.

Note that Greek indices μ, ν will run over time and space both, whereas i, j will run only over the three spatial coordinates, and thus

$$x^\mu = (x^0, x^i) \quad (1.2)$$

I will sometimes use x^0 to denote the time component and sometimes x^t , depending on what I feel looks better in that particular formula.

I will always set $\hbar = c = 1$.² They can be restored if required from dimensional analysis.

1.3 Quantum mechanics review

In QFT I so far, you have understood in great detail how to compute things in quantum *mechanics* using the path-integral; in other words you studied quantum mechanical systems with a classical real-time Lagrangian

$$L(q, \dot{q}) = \frac{1}{2}\dot{q}^2 - V(q) \quad S = \int dt L(q, \dot{q}) \quad (1.3)$$

You then inserted this into a path-integral to define the *generating functional*, which was

$$Z[J] \equiv \mathcal{N} \int [\mathcal{D}q] \exp \left(iS[q] + i \int dt J(t)q(t) \right) \quad (1.4)$$

where $\int [\mathcal{D}q]$ denotes the philosophically soothing but mathematically somewhat distressing “integral over all functions”. In the previous QFT1 course, you learned that to compute the *time-ordered Green’s function*, you simply had to bring down factors of $q(t)$ when doing the average, i.e.

$$\langle 0 | \hat{q}(t_N) \hat{q}(t_{N-1}) \cdots \hat{q}(t_1) | 0 \rangle = \mathcal{N} \int [\mathcal{D}q] q(t_N) q(t_{N-1}) \cdots q(t_1) \exp(iS[q]) \quad t_N > t_{N-1} > \cdots t_1 \quad (1.5)$$

Note it is absolutely crucial here that on the left-hand side $t_N > t_{N-1} > \cdots t_1$. On the right-hand side it doesn’t seem to matter; on the left-hand side it does, because these are quantum operators that do not commute. It was explained in QFT I that path integrals only give you correlation functions that are time-ordered; to emphasize this, I will often put a T around the correlation function, i.e. I may write the left-hand side as

$$\langle 0 | T(\hat{q}(t_N) \hat{q}(t_{N-1}) \cdots \hat{q}(t_1)) | 0 \rangle \quad (1.6)$$

In fact, this quantity was typically somewhat ill-defined, even by physicist standards. However you could make it better by *Wick-rotating*, i.e. you wrote

$$t = -i\tau \quad (1.7)$$

where τ is *real*. This is sometimes called “Euclidean time”; note that the metric becomes

$$ds^2 = dt^2 - d\vec{x}^2 = -(d\tau^2 + d\vec{x}^2). \quad (1.8)$$

² k_B makes an appearance in a homework problem (or rather it would, if I hadn’t set it to 1).

The overall minus sign here is of no importance; we are working in a spacetime with Euclidean signature. Note that the action also becomes

$$iS = - \int d\tau \left(\frac{1}{2} \left(\frac{dq}{d\tau} \right)^2 + V(q) \right) \equiv -S_E \quad (1.9)$$

where S_E is positive-definite (if the potential is positive-definite). This Wick-rotated path integral is

$$Z[J] \equiv \mathcal{N} \int [\mathcal{D}q] \exp \left(-S_E + \int d\tau J(\tau)q(\tau) \right) \quad (1.10)$$

Note that the object in the exponential is now real rather than imaginary, and this makes the integral convergent³. In general, many confusions about path integrals can be made to go away by imagining a Wick-rotation to Euclidean signature. As explained in the first part, this prescription also guarantees that you are in the vacuum of the theory by suppressing the contribution of all states with energy E by a factor of $\exp(-E\tau)$.

2 Path Integrals in Free Quantum Field Theory

With this under our belt, we will now move on to quantum field theory. You have studied the basic ideas in IFT; let me just remind you. We will begin with a study of the free real scalar field ϕ . This has action

$$S[\phi] = \frac{1}{2} \int d^4x \left((\partial\phi(x))^2 - m^2\phi(x)^2 \right) = \frac{1}{2} \int d^4x \phi(x)(-\partial^2 - m^2)\phi(x) \quad (2.1)$$

The classical equations of motion arising from the variation of this action are

$$(\partial^2 + m^2)\phi(x) = 0 \quad (2.2)$$

This is philosophically the same as the quantum mechanics problem studied earlier. To make the transition imagine going from the single quantum-mechanical variable q to a large vector q_a where a runs (say) from 1 to N . Now imagine formally that a runs over all the sites of a lattice that is a discretization of space, and now $q_a(t)$ is basically the same thing as $\phi(x^i; t)$ which is exactly the system we are studying above.

Now we would like to study the quantum theory. First, I point out that we can define a path integral in precisely the same way as before, i.e. we can consider the following path-integral:

$$Z \equiv \int [\mathcal{D}\phi] \exp(iS[\phi]) \quad (2.3)$$

Where $S[\phi]$ is the action written down above, and $[\mathcal{D}\phi]$ now represents the functional integral over all *fields* and not just particle trajectories.

There are two main things that are nice about doing quantum field theory from path integrals the way discussed above. One of them is honest, the other is a bit “secret”.

1. The honest one: all of the symmetries of the problem are manifest. The action S is Lorentz-invariant, and it is fairly easy to see how these symmetries manifest themselves in a particular computation. Compare this to the Hamiltonian methods used in IFT, where you have to pick a time-slice and it always seems like a miracle when final answers are Lorentz-invariant.
2. The secret one: the path integral allows one to be quite cavalier about subtle issues like “what is the structure of the Hilbert space exactly”. This is very convenient when we get to gauge fields, where there are subtle constraints in the Hilbert space (google “Dirac bracket”) that you can more or less not worry about when using the path integral (i.e. one can go quite far in life without knowing exactly what a “Dirac bracket” is).

³Well, more convergent than before, at least.

2.1 The generating functional

Now that the philosophy is out of the way, let us do a computation. We will begin by computing the following two-point function:

$$\langle 0|T(\phi(x)\phi(y))|0\rangle \quad (2.4)$$

This object is called the *Feynman propagator*. It is quite important for many reasons; I will discuss them later, for now let's just calculate it. By arguments identical to those leading to (1.5), we see that we want

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = Z_0^{-1} \int [\mathcal{D}\phi] \phi(x)\phi(y) \exp(iS[\phi]) \quad (2.5)$$

To calculate this, it is convenient to define the same generating functional as we used for quantum mechanics

$$Z[J] \equiv \int [\mathcal{D}\phi] \exp\left(iS[\phi] + i \int d^4x J(x)\phi(x)\right) \quad (2.6)$$

And we then see from functional differentiation that the two-point function is

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \frac{1}{Z_0} \left(-i \frac{\delta}{\delta J(x)}\right) \left(-i \frac{\delta}{\delta J(y)}\right) Z[J] \Big|_{J=0} \quad (2.7)$$

Each functional derivative brings down a $\phi(x)$. Now we will evaluate this function. We first note the identity derived in QFT I for doing Gaussian integrals in Appendix A of [5], which I have embellished with a few i 's and factors of 2 here and there:

$$\int dx_1 dx_2 \cdots dx_N \exp\left(-\frac{1}{2} x^a A_{ab} x^b + i J_a x^a\right) = \sqrt{\frac{(2\pi)^N}{\det A}} \exp\left(-\frac{1}{2} J_a (A^{-1})^{ab} J_b\right) \quad (2.8)$$

We note from the form of the action (2.1) that the path integral $Z[J]$ we want to do is of precisely this form, where we do our usual “many integrals” limit and where a labels points in space and the operator A is

$$A = i(\partial^2 + m^2) \quad (2.9)$$

We conclude that the answer for $Z[J]$ is

$$Z[J] = \det\left(\frac{\partial^2 + m^2}{-2\pi i}\right)^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \int d^4x d^4y J(x) D_F(x, y) J(y)\right) \quad (2.10)$$

where I have given the object playing the role of A^{-1} a prescient name D_F . It is the inverse of the differential operator defined in (2.9) and thus satisfies

$$i(\partial^2 + m^2) D_F(x, y) = \delta^{(4)}(x - y) \quad (2.11)$$

This is an important result. Let us first note that the path integral is asking us to compute the *functional determinant* of a differential operator. This is a product over infinitely many eigenvalues; it is quite a beautiful thing but we will not really need it here, so we will return to it later.

The next thing to note is that the dependence on J is quite simple; the exponential of a quadratic. Indeed, inserting this into (2.7) we get

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = D_F(x, y) \quad (2.12)$$

Thus, we have *derived* that the time-ordered correlation function of $\phi(x)$ is given by the inverse of $i(\partial^2 + m^2)$. You have already encountered this phenomenon in IFT.

2.2 Calculating the Feynman propagator: poles, analytic continuation and $i\epsilon$

Let us now actually calculate this object. I note that you have performed a similar computation in IFT, but I will repeat some elements of it to explain what the path integral is doing for us. We first go to Fourier space:

$$D_F(x, y) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot (x-y)} \tilde{D}_F(p) \quad (2.13)$$

Inserting this into (2.11) we find

$$\int \frac{d^4 p}{(2\pi)^4} (-p^2 + m^2) e^{-ip \cdot (x-y)} \tilde{D}_F(p) = -i\delta^{(4)}(x-y) \quad (2.14)$$

We now see that we want the object in momentum space $\tilde{D}_F(p)$ to be proportional to $p^2 - m^2$ so that we can use the identity $\int d^4 p e^{ip \cdot x} = (2\pi)^4 \delta^{(4)}(x)$. Getting the factors right, we find the following expression for the propagator in Fourier space:

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2} \quad (2.15)$$

This looks nice, but actually, this expression is not yet complete, in that specifying this Fourier transform does *not* yet completely specify a function $D_F(x, y)$ in position space. To understand this, let us actually attempt to Fourier transform this thing back. We break the integral into time p_0 and spatial \vec{p} and do the time integral first. Let me define $\omega_{\vec{p}} \equiv +\sqrt{\vec{p}^2 + m^2}$:

$$D_F(x, y) = \int \frac{dp_0}{2\pi} \int \frac{d^3 p}{(2\pi)^3} \frac{i}{p_0^2 - p^2 - m^2} e^{-ip_0(x^0 - y^0) + i\vec{p} \cdot (\vec{x} - \vec{y})} \quad (2.16)$$

$$= i \int \frac{dp_0}{2\pi} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p_0 + \omega_{\vec{p}}} \frac{1}{p_0 - \omega_{\vec{p}}} e^{-ip_0(x^0 - y^0) + i\vec{p} \cdot (\vec{x} - \vec{y})} \quad (2.17)$$

Now if we look at the integral, we see that there are poles when $p_0 = \pm\omega_{\vec{p}}$. The correct way to think about this is to imagine it as an integral in the complex p_0 plane, and we then have a pole. To actually perform the integral, we need to specify *how we go around the poles*; different ways of going around the poles will give us different answers.

There are two steps; first let's say that $x^0 - y^0 > 0$; in the case we must complete the contour below the first pole for the integral to converge. We are not done yet; to get the second bit of information, we should recall that path integrals only make sense if we formulate them in Euclidean signature. This helps; let's imagine that we formulated the whole thing in Euclidean time all along. Recall from (1.7) that the Wick rotation to Euclidean time and frequency is

$$t = -i\tau \quad p_0 = ip_0^E \quad (2.18)$$

This means that if we were working in Euclidean time all along, we would have done the p_0 integral up and down the imaginary axis. This tells us which way to complete the integral, as only one pole prescription (one up, one down) is compatible with this.

Now we can do the integral. We pick up the contribution from the $+\omega_{\vec{p}}$ when $x^0 - y^0 > 0$, and the answer is

$$D_F(x, y) = \frac{(-2\pi i)i}{2\pi} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} e^{-i\omega_{\vec{p}}(x^0 - y^0) + i\vec{p} \cdot (\vec{x} - \vec{y})} \quad x_0 > y_0 \quad (2.19)$$

Similarly, if $x_0 < y_0$ we close the integral the other way and find

$$D_F(x, y) = \frac{(2\pi i)i}{2\pi} \int \frac{d^3 p}{(2\pi)^3} \frac{1}{-2\omega_{\vec{p}}} e^{+i\omega_{\vec{p}}(x^0 - y^0) + i\vec{p} \cdot (\vec{x} - \vec{y})} \quad x_0 < y_0 \quad (2.20)$$

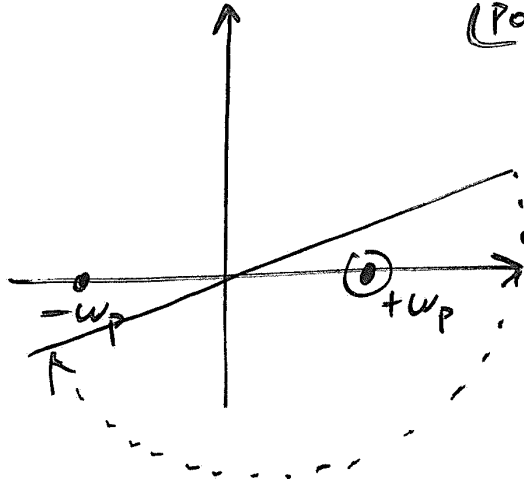


Figure 1: Circling poles in the complex frequency plane.

This is it: we have done the hard work in calculating the time-ordered correlation function, also called the *Feynman propagator*. The spatial integral can be done, but it involves a bit of work with Bessel functions and I may assign it as a homework.

A convenient way to summarize this business with the pole is to write the Green's function in momentum space as

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2 + i\epsilon} \quad (2.21)$$

where ϵ is a tiny number that tips the contour slightly upwards. This is called the *Feynman prescription*.

To summarize: path integrals give us time-ordered correlation functions. If ever we are confused about how to go around poles, then we should remember that the path integral secretly only makes sense in Euclidean signature; this typically helps.

2.3 Higher-point functions in the free theory

We just computed the two-point functions. However, one can of course compute higher point functions using just the same idea: e.g. say you want

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle \quad (2.22)$$

Just as in (2.7), we determine this by taking functional derivatives of $Z[J]$:

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = (-i)^4 \frac{1}{Z_0} \prod_{i=1}^4 \frac{\delta}{\delta J(x_i)} Z[J] \quad (2.23)$$

However, because $Z[J]$ is Gaussian, the resulting answer is just different combinations of $D_F(x, y)$, e.g. you can work out:

$$\langle 0|T\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)|0\rangle = (-i)^4 (D_F(x_1, x_2)D_F(x_3, x_4) + D_F(x_1, x_3)D_F(x_2, x_4) + D_F(x_1, x_4)D_F(x_2, x_3)) \quad (2.24)$$

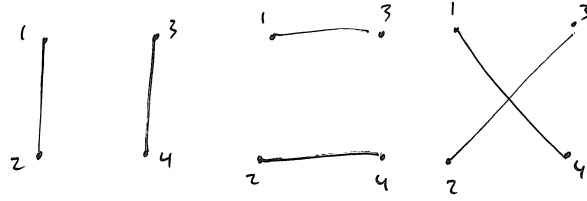


Figure 2: Different ways to draw propagators in Wick's theorem.

In other words, you draw propagators between all possible pairs of points. It should be clear that this works for any number of insertions; this is called *Wick's theorem*.

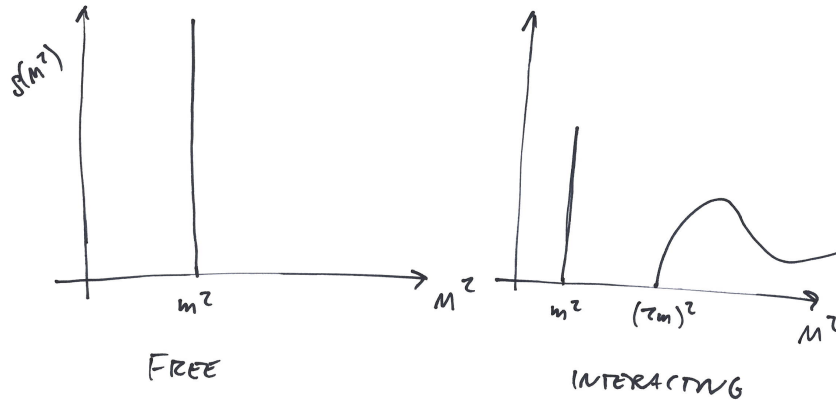


Figure 3: Schematic plots of the spectral representation in interacting theories.

3 Interacting theories

Next, we would like to study interacting quantum field theories. You have encountered an example of one such already, the $\lambda\phi^4$ theory:

$$S[\phi] = \int d^4x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \right) \quad (3.1)$$

This describes a field theory where the particles that are captured by ϕ can interact or *scatter* off of each other, and where roughly speaking the probability for this scattering to occur is given by λ .

The first thing to note is that in general *we simply cannot solve interacting field theories*. In this course we will instead use the fact that we *can* solve free quantum field theories and try to solve interacting quantum field theories order by order in the interaction strength (i.e. in λ above). This is called perturbation theory. Sometimes it is a good guide to the physics, and sometimes it isn't.

Before plunging in, let us understand the sorts of things that can happen in interacting quantum field theory by considering again our friend $\langle T\phi(x)\phi(y) \rangle$. Denote by $D_F(x, y; M^2)$ the Feynman propagator for a *free* scalar field of mass M ; then it turns out (see Chapter 7.1 of [1]) that in *any* Lorentz-invariant theory we can write the 2-point function in the *Kallen-Lehmann spectral representation* as:

$$\langle T\phi(x)\phi(y) \rangle = \int_0^\infty \frac{dM^2}{2\pi} \rho(M^2) D_F(x, y; M^2) \quad (3.2)$$

$\rho(M^2)$ is called the “spectral density” and roughly tells you “what are the masses of states that ϕ is creating from the vacuum?” For example, if we were doing a free scalar field, then we would find that

$$\rho_{\text{free}}(M^2) = 2\pi\delta(m^2 - M^2) \quad (3.3)$$

i.e. we create just a single massive state, which is the single particle state of a single ϕ particle. On the other hand, imagine that you study an interacting theory, e.g. the $\lambda\phi^4$ theory above: we then get something much more interesting, e.g. as below: This takes the form

$$\rho_{\text{interacting}}(M^2) = 2\pi Z\delta(m^2 - M^2) + \text{other things} \quad (3.4)$$

Note that the *weight* of the delta function has changed: as you now have some other probability to make other things, the probability to create the single-particle state has been reduced.

Note that whenever Z is finite, it is at least true that we still create a particle. What if Z drops to zero? Then there is no probability that the ϕ field will create a physical particle. This is what happens in QCD due to confinement, as you will see in due course.

3.1 LSZ Reduction Formula

From now on we assume that the Z appearing in the formula above is finite. We would now like to understand how to compute scattering amplitudes: e.g. we would like to understand the question: if you smash together two protons very hard, then what is the probability that a Higgs boson comes out? To do this we need to make a connection between matrix elements of “single-particle states” (e.g. 2 protons, 1 Higgs, etc.) and things that we know how to compute (i.e. time-ordered correlation functions). This connection happens through the LSZ Reduction Formula. My treatment follows that in Chapter 5 of [4].

I will work everything out in an example from the theory above, and we will consider doing $2 - 2$ scattering. So we need to prepare an initial and a final state. As we are discussing states, let me now return to the canonical formalism, where we have in the Heisenberg picture:

$$\phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2\omega_{\vec{p}}} \left(a_{\vec{p}} e^{-ip \cdot x} + a_{\vec{p}}^\dagger e^{+ip \cdot x} \right) \quad p_\mu = (\omega_{\vec{p}}, \vec{p}) \quad (3.5)$$

It is easy enough to solve this for $a_{\vec{p}}$ to get:

$$a_{\vec{p}} = \int d^3x e^{ip \cdot x} (i\partial_t \phi(x) + \omega_{\vec{p}} \phi(x)) \quad (3.6)$$

$$= i \int d^3x e^{ip \cdot x} \overleftrightarrow{\partial}_t \phi(x) \quad (3.7)$$

where this two-headed arrow represents $f \overleftrightarrow{\partial} g = f \partial g - (\partial f) g$. Note that in an interacting theory this a depends on time.

For technical reasons, we will want to have wave-packets, i.e. we imagine some momentum envelope

$$f_1(\vec{k}) \sim \exp\left(-\frac{(\vec{k} - \vec{k}_1)^2}{4\sigma^2}\right) \quad (3.8)$$

and we then construct an operator that is a momentum-smearred version of the usual creation operator:

$$a_1^\dagger \equiv \int d^3k f_1(\vec{k}) a_{\vec{k}}^\dagger \quad (3.9)$$

Now in the free theory, we *know* that we create an initial state by acting with the creation operators. We will simply *assume* that in the interacting theory we create an initial state by acting with the creation operator in the distant past, and that this does what we want up to a renormalization factor Z : i.e. to make an initial state of one particle in a wave-packet around \vec{k}_1 , we write

$$|\text{initial packet}; \vec{k}_1\rangle = \frac{1}{\sqrt{Z}} \lim_{t \rightarrow -\infty} a_1^\dagger |0\rangle \quad (3.10)$$

It may not be clear, but this factor Z is the same as the one that I introduced earlier. We will be interested in an initial state of say two particles:

$$|i\rangle = \lim_{t \rightarrow -\infty} \frac{1}{Z} a_1^\dagger a_2^\dagger |0\rangle \quad (3.11)$$

Similarly, the final state is made out of operators that are defined at $t = +\infty$:

$$|f\rangle = \lim_{t \rightarrow +\infty} \frac{1}{Z} a_1^\dagger a_2^\dagger |0\rangle \quad (3.12)$$

The amplitude that we are looking for is $\langle f|i\rangle$, i.e.

$$\langle f|i\rangle = \left(\frac{1}{\sqrt{Z}}\right)^4 \langle 0|T a_{1'}(+\infty) a_{2'}(+\infty) a_1^\dagger(-\infty) a_2^\dagger(-\infty)|0\rangle \quad (3.13)$$

Note that everything is time-ordered anyway; thus we can stick a “T” for time-ordering in there with no issues.

I now want to express this amplitude in terms of the field $\phi(x)$. To do this, let us first derive an equation relating $a_1(t = +\infty)$ to $a_1(t = -\infty)$. We write

$$a_1^\dagger(+\infty) - a_1^\dagger(-\infty) = \int_{-\infty}^{+\infty} dt \partial_t a_1^\dagger(t) \quad (3.14)$$

$$= \int d^3k f_1(\vec{k}) \int d^4x \partial_t (e^{-ik \cdot x} (-i\partial_t \phi(x) + \omega_{\vec{k}} \phi(x))) \quad (3.15)$$

$$= -i \int d^3k f_1(\vec{k}) \int d^4x e^{-ik \cdot x} (\partial_t^2 + \omega_{\vec{k}}^2) \phi(x) \quad (3.16)$$

$$= -i \int d^3k f_1(\vec{k}) \int d^4x e^{-ik \cdot x} (\partial_t^2 + \vec{k}^2 + m^2) \phi(x) \quad (3.17)$$

$$= -i \int d^3k f_1(\vec{k}) \int d^4x e^{-ik \cdot x} (\partial_t^2 - \nabla^2 + m^2) \phi(x) \quad (3.18)$$

$$= -i \int d^3k f_1(\vec{k}) \int d^4x e^{-ik \cdot x} (\partial^2 + m^2) \phi(x) \quad (3.19)$$

Here we use the definition of the creation operator from the field; then we use the Fourier transform to trade the momentum for a spatial gradient, and then use the fact that the wave-packets are localized to integrate by parts with impunity. We end with the wave operator acting on the field. Note that for a free field this is zero by the equations of motion; for an interacting field it is *not*. We conclude with the following formula

$$a_1^\dagger(-\infty) = a_1^\dagger(+\infty) + i \int d^3k f_1(\vec{k}) \int d^4x e^{-ik \cdot x} (\partial^2 + m^2) \phi(x) \quad (3.20)$$

There is a corresponding expression for the annihilation operator:

$$a_1(+\infty) = a_1(-\infty) + i \int d^3k f_1(\vec{k}) \int d^4x e^{+ik \cdot x} (\partial^2 + m^2) \phi(x) \quad (3.21)$$

Now insert these expressions into (3.13): here we see the magic of time-ordering: every $a(-\infty)$ is sent to the right and annihilated against the vacuum ket. Similarly every $a^\dagger(+\infty)$ is sent to the *left* and annihilated against the vacuum bra. The wave packets have done their magic; we now set them all equal to delta functions $f_1(\vec{k}) = \delta^{(3)}(\vec{k} - \vec{k}_1)$. Thus the full answer for the amplitude (where I generalize now to n in-particles and n' out-particles) becomes

$$\langle f|i\rangle = i^{n+n'} \int d^4x_1 Z^{-\frac{1}{2}} e^{-ik_1 \cdot x_1} (\partial_1^2 + m^2) \cdots d^4x_{1'} e^{+ik_{1'} \cdot x_{1'}} Z^{-\frac{1}{2}} (\partial_{1'}^2 + m^2) \cdots \langle 0|T(\phi(x_1)\phi(x_2)\cdots\phi(x'_1)\phi(x'_2)\cdots)|0\rangle \quad (3.22)$$

This is the LSZ reduction formula, relating matrix elements to time-ordered n -point functions. Let’s express it in momentum space, where it is:

$$\prod_{i=1}^n \frac{i\sqrt{Z}}{k_i^2 - m^2} \prod_{i=1}^{n'} \frac{i\sqrt{Z}}{k_{i'}^2 - m^2} \langle f|i\rangle = \langle 0|\phi(k_1)\phi(k_2)\cdots\phi(k'_n)|0\rangle \quad (3.23)$$

where the right-hand side is understood to be the Fourier transform of the time-ordered correlation function. In words: to calculate the transition amplitude (which is the physical thing), you should Fourier transform

the correlation function, and then put all the momentum close to their “on-shell values” (i.e. $k_i^2 \sim m^2$). In that kinematic regime the correlation function should develop a pole so that it looks like the left-hand side; the *coefficient* of that pole is the transition amplitude you are looking for.

Whew. We now move on to actually calculating the multi-point function.

3.2 Perturbation theory and Feynman diagrams

We now understand that to calculate transition amplitudes, we should compute correlation functions in interacting field theories, e.g. this one:

$$S[\phi] = \int d^4x \left(-\frac{1}{2}(\partial\phi)^2 - \frac{m^2}{2}\phi^2 - \frac{\lambda}{4!}\phi^4 \right) \quad (3.24)$$

From the previous section, we understand that if we can compute the generating functional $Z[J]$

$$Z[J] \equiv \int [\mathcal{D}\phi] \exp \left(iS[\phi] + i \int d^4x J(x)\phi(x) \right) \quad (3.25)$$

and take functional derivatives then we will obtain the required correlation functions. There is no hope of doing this exactly for all λ ; instead we will expand order by order in λ . The first thing to do is to expand the action as

$$S[\phi] = S_0[\phi] + \int d^4x \mathcal{L}_1(\phi(x)) \quad (3.26)$$

with S_0 the free action and \mathcal{L}_1 the interacting part: in our case $\mathcal{L}_1(\phi) = \frac{\lambda}{4!}\phi^4$. Now we will do a trick; notice that every time you have a ϕ , you can replace it with a functional derivative with respect to J . Thus we can write

$$Z[J] = \int [\mathcal{D}\phi] \exp \left(iS_0[\phi] + i \int d^4x (\mathcal{L}_1(\phi(x)) + J(x)\phi(x)) \right) \quad (3.27)$$

$$= \int [\mathcal{D}\phi] \exp \left(i \int d^4x \mathcal{L}_1 \left(-i \frac{\delta}{\delta J(x)} \right) \right) \exp (iS_0[\phi] + iJ(x)\phi(x)) \quad (3.28)$$

$$= \exp \left(i \int d^4x \mathcal{L}_1 \left(-i \frac{\delta}{\delta J(x)} \right) \right) Z_0[J] \quad (3.29)$$

where $Z_0[J]$ is the generating functional for the *free* quantum field theory. We calculated this before. We find then

$$Z[J] = Z_0[0] \exp \left(i \int d^4x \mathcal{L}_1 \left(-i \frac{\delta}{\delta J(x)} \right) \right) \exp \left(-\frac{1}{2} \int d^4x d^4y J(x) D_F(x, y) J(y) \right) \quad (3.30)$$

Here I have absorbed the functional determinant into the object $Z_0[0]$.

In particular, if we specialize to the $\lambda\phi^4$ theory, then we can write \mathcal{L}_1 explicitly as

$$Z[J] \propto \exp \left(i \int d^4x \frac{\lambda}{4!} \left(-i \frac{\delta}{\delta J(x)} \right)^4 \right) \exp \left(-\frac{1}{2} \int d^4x d^4y J(x) D_F(x, y) J(y) \right) \quad (3.31)$$

We can now work out $Z[J]$ by taking many many functional derivatives. This turns out to organize itself into the delightful structure of Feynman diagrams. To understand this, we basically just expand everything:

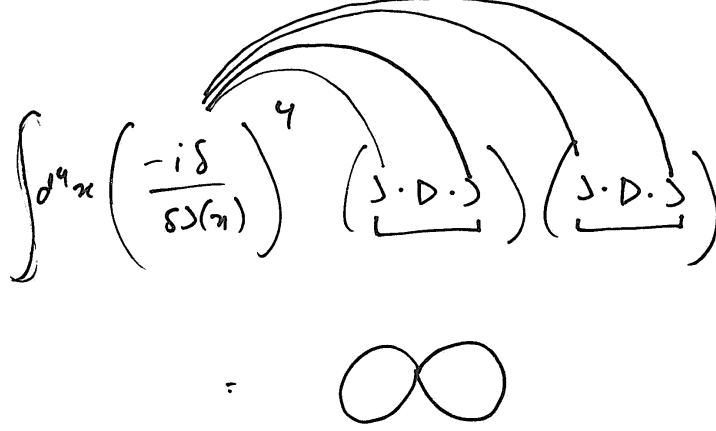


Figure 4: Picture of how functional derivatives act and representation in terms of Feynman diagram.

expand \mathcal{L}_1 in powers of the coupling λ and expand the free partition function in powers of the source J :

$$\exp\left(-i \int d^4 x \frac{\lambda}{4!} \left(\frac{-i\delta}{\delta J(x)}\right)^4\right) = 1 - i \int d^4 x \frac{\lambda}{4!} \left(\frac{-i\delta}{\delta J(x)}\right)^4 + \frac{1}{2} \left(i \int d^4 x \frac{\lambda}{4!} \left(\frac{-i\delta}{\delta J(x)}\right)^4\right)^2 + \dots \quad (3.32)$$

$$\exp\left(-\frac{1}{2} J \cdot D \cdot J\right) = 1 - \frac{1}{2} J \cdot D \cdot J + \frac{1}{2} \left(-\frac{1}{2} J \cdot D \cdot J\right)^2 + \dots \quad (3.33)$$

where for a shorthand I am denoting

$$J \cdot D \cdot J \equiv \int d^4 x d^4 y J(x) D_F(x, y) J(y) \quad (3.34)$$

Now we work out the product. I first work out an example to demonstrate the structure, and I then describe the systematic rules for doing this. Examine the first non-trivial term:

$$-i \int d^4 x \frac{\lambda}{4!} \left(\frac{-i\delta}{\delta J(x)}\right)^4 \frac{1}{2} \left(-\frac{1}{2} J \cdot D \cdot J\right) \left(-\frac{1}{2} J \cdot D \cdot J\right) \quad (3.35)$$

Now each of the functional derivatives acts on one of the J 's, and the D_F 's tie together the two J 's that it is attached to. We end up with something that looks like

$$-\frac{N}{4!} i \lambda \int d^4 x D_F(x, x) D_F(x, x) \quad (3.36)$$

Where it is convenient to capture this with a picture, called a *Feynman diagram*, where a line indicates a propagator from x to y and a vertex indicates an insertion of the interaction λ . In a second we will discuss the numerical prefactor N . First let's try to understand how this works for the next term,

$$i \int d^4 x d^4 y \frac{\lambda}{4!} \left(\frac{-i\delta}{\delta J(x)}\right)^4 \frac{\lambda}{4!} \left(\frac{-i\delta}{\delta J(y)}\right)^4 \frac{1}{4!} \left(-\frac{1}{2} J \cdot D \cdot J\right)^4 \quad (3.37)$$

The rules should be clear: assemble four propagators and two vertices however you can. There are three topologically distinct terms that appear, and they look like this:

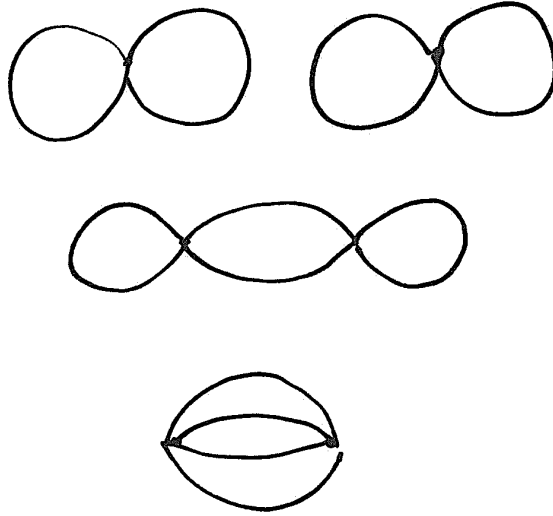


Figure 5: Three different diagrams that contribute at order λ^2 .

Finally, I should stress that there are of course diagrams that have J 's left over, e.g.

$$-i \int d^4x \frac{\lambda}{4!} \left(\frac{-i\delta}{\delta J(x)} \right)^4 \frac{1}{4!} \left(-\frac{1}{2} J \cdot D \cdot J \right)^4 \quad (3.38)$$

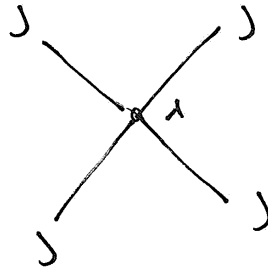


Figure 6: Diagram with external J 's.

Intuitively, you should imagine these pictures as representing movements of particles in space and time; they can annihilate and recombine at vertices. Feynman diagrams are the basic tool in particle physics. By now it should be clear that there are *rules* that give you the amplitude associated with each Feynman diagram. In position space they are very simple:

- Each vertex comes with a factor of $-i \int d^4x \lambda$.
- Each propagator comes with a factor of $D_F(x, y)$.
- Each external J comes with a $\int d^4x J(x)$.

We will shortly rewrite them in momentum space, as they are slightly nicer in that form.

3.2.1 Combinatorics and symmetry factors

Now let's examine the combinatorics more carefully. From the double expansion in (3.31), we see that a particular term takes the form

$$\sum_{V,P=0}^{\infty} \frac{1}{V!} \left[\left(\int d^4x \frac{-i\lambda}{4!} \left(-i \frac{\delta}{\delta J(x)} \right)^4 \right) \right]^V \frac{1}{P!} \left[-\frac{1}{2} J \cdot D \cdot J \right]^P \quad (3.39)$$

where V is the number of vertices and P the number of propagators. Now when you draw a picture, e.g. **two kissing bubbles**, this represents several different ways of contracting functional derivatives against sources: in other words, the same term appears a number of different times. How many? Well, there are $4!$ different ways to rearrange the derivatives coming out of a vertex, there are $P!$ ways to rearrange the P propagators amongst themselves, there are 2 ways to pick the two sides of a propagator, and there $V!$ ways to rearrange the vertices. These factors precisely cancel all the factors in the denominator – and thus one might naively think that there is thus no combinatorics to do at all. (Note this is why the funny $4!$ is in the definition of the interaction vertex as $\frac{\lambda}{4!}$)

However this is not true, because we have actually overcounted. In doing that counting, we assumed that every rearrangement of propagators and vertices would give a distinct contribution: in reality this is not true, as they can “cancel” each other. For example, in **two-kissing bubbles**, note that if we switch the two sides of the propagator and also switch the two corresponding functional derivatives at the vertex, then we get the same contribution again. Similarly for the other side, and also for the two propagators themselves. Thus we have overcounted by a factor of $2 \cdot 2 \cdot 2 = 8$, and we must divide by this: thus the full diagram in this case is

$$\frac{i\lambda}{8} \int d^4x D_F(x, x) D_F(x, x) \quad (3.40)$$

This factor of 8 is called the *symmetry factor* S of the diagram, as it always has to do with some operation that leaves the diagram invariant.

It is a fun exercise in pure thought to figure out the symmetry factor; I recommend doing it for several other diagrams to get the hang of it. To summarize: to determine the right prefactor, figure out the correct symmetry factor and divide by it.

3.3 Connected diagrams

We now study a way to re-organize this sum over diagrams. Note that the generic contribution to $Z[J]$ is a sum over several disconnected diagrams (where “disconnected” is the intuitively obvious “I can draw a line through it and separate it into two pieces”). If I label each *connected* diagram by I , call the value of the diagram C_I , then the generic *disconnected* diagram gives a contribution

$$D = \frac{1}{S_D} \prod_I (C_I)^{n_I} \quad (3.41)$$

where n_I is the number of times each connected diagram appears and S_D now being a *new* symmetry factor associated with rearranging the connected diagrams themselves. As we have already dealt with the rearrangements *inside* each connected diagram, S_D discusses rearrangements of each connected diagram as a whole, and so is always

$$S_D = \prod_I n_I! \quad (3.42)$$

Thus we have

$$\begin{aligned} Z[J] &= \sum_{\{n_I\}} \prod_I \frac{1}{n_I!} (C_I)^{n_I} \\ &= \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!} (C_I)^{n_I} \\ &= \prod_I \exp(C_I) \\ &= \exp\left(\sum_I C_I\right) \end{aligned} \tag{3.43}$$

This is very nice – it says that the total $Z[J]$ is the exponential of the sum over only *connected* diagrams. It's convenient to define the logarithm of the generating functional $W[J]$ as this sum.

$$\boxed{iW[J] = \sum_I C_I \quad Z[J] = \exp(iW[J])} \tag{3.44}$$

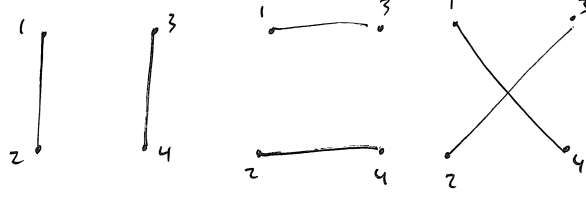


Figure 7: Wick's theorem figure, repeated here to illustrate λ^0 contribution to 2-2 scattering: replace 3 with 1' and 4 with 2'.

3.4 Scattering amplitudes

Now that we have all of this technology, let's finally *use it*. Imagine in this $\lambda\phi^4$ theory we want to calculate the 2 – 2 scattering of two particles. In other words, we want to compute the amplitude:

$$\langle f|i \rangle = \langle k'_1, k'_2; \text{out} | k_1, k_2; \text{in} \rangle \quad (3.45)$$

From the LSZ formula we see that this is equal to:

$$\langle i|f \rangle = \left(\frac{i}{\sqrt{Z}} \right)^{n+n'} \int d^4x_1 e^{-ik_1 \cdot x_1} \dots d^4x_{1'} e^{+ik_{1'} \cdot x_{1'}} \dots \langle 0|T\phi(x_1)\phi(x_2)\phi(x'_1)\phi(x'_2)|0 \rangle (k_1^2 - m^2) \dots (k_{1'}^2 - m^2) \quad (3.46)$$

This is simply (3.23) with the rhs written as a Fourier transform of the position space correlator.

So now we need to calculate the four point function, which I will abbreviate as $G^{(4)}(\{x_i\})$. If I also further abbreviate

$$\delta_i \equiv -i \frac{\delta}{\delta J(x_i)} \quad (3.47)$$

Then the four-point function is given by

$$G^{(4)}(\{x_i\}) = \frac{1}{Z[J]} \delta_1 \delta_2 \delta_{1'} \delta_{2'} Z[J] \Big|_{J=0} \quad (3.48)$$

As there are four functional derivatives acting on $Z[J]$, we should write down the terms in the diagrammatic expansion of $Z[J]$ that contain four external J 's, and then taking the functional derivative simply associates each external J with an external insertion. We do this order by order: at order λ^0 we find the terms in Figure 7.

Let's examine the Fourier transform of one of these terms, e.g. the first one is

$$\int d^4x_1 e^{-ik_1 \cdot x_1} \dots d^4x_{1'} e^{+ik_{1'} \cdot x_{1'}} D_F(x_1, x'_1) D_F(x_2, x'_2) \quad (3.49)$$

But by translational invariance, the propagator only cares about the *difference* between the two points, so this can be rewritten as

$$\int d^4x_1 e^{-ik_1 \cdot x_1} \dots d^4x_{1'} e^{+ik_{1'} \cdot x_{1'}} D_F(x_1 - x'_1) D_F(x_2 - x'_2) \quad (3.50)$$

Now notice that the sum $x_1 + x'_1$ does not enter in the expression with the propagators. I can rearrange the thing in the exponent of the Fourier transform to read

$$k_{1'} \cdot x_{1'} - k_1 \cdot x_1 = \frac{1}{2}(k_1 - k_{1'}) \cdot (x_1 + x'_1) + \frac{1}{2}(k_1 + k_{1'}) \cdot (x_1 - x'_1) \quad (3.51)$$

Now the integral over the combination $x_1 + x'_1$ is easy to do, as it does not enter into the integrand. This integral makes the whole thing proportional to a delta function

$$\delta^{(4)}(k_1 - k'_1) \quad (3.52)$$

But this means that the incoming momentum is equal to the outgoing momentum: so there is no scattering! This should make intuitive sense from the picture: the particle 1 is becoming the 1' without doing anything. Similar considerations apply to the other disconnected diagrams.

In general, scattering will only occur from *connected* diagrams; from the discussion above, this means that in the LSZ formula we should actually use only derivatives of $iW[J]$, i.e.

$$G^{(4)}(\{x_i\})_C = \delta_1 \delta_2 \delta_{1'} \delta_{2'} iW[J] \Big|_{J=0} \quad (3.53)$$

$W[J]$ is trivial to lowest order in λ : so we see that there is no scattering at order λ^0 , which makes sense.

We now go to the next order in λ . Here we find one contribution that goes like **picture of cross diagram**. This will contribute non-trivially to scattering: in position space, this turns out to be

$$G^{(4)}(\{x_i\})_C = -i\lambda \int d^4y D_F(x_1 - y) D_F(x_2 - y) D_F(x'_1 - y) D_F(x'_2 - y) \quad (3.54)$$

It is very useful to go to momentum space. We Fourier transform everything using the Fourier representation of the propagator:

$$D_F(x - y) = \int \frac{d^4p}{(2\pi)^4} e^{\pm ip \cdot (x-y)} \frac{i}{p^2 - m^2} \quad (3.55)$$

In this expression it is up to us how to pick the momentum in the exponent: to line up with the inverse Fourier transform in the LSZ formula it is convenient to pick the positive exponent for the *in* states and negative one for the *out* states. Thus I find

$$G^{(4)}(\{x_i\})_C = -i\lambda \int d^4y \prod_{i=1}^4 \frac{d^4p_i}{(2\pi)^4} \exp^{ip_1(x_1-y) + ip_2(x_2-y) - ip'_1(x'_1-y) - ip'_2(x'_2-y)} \frac{i}{p_1^2 - m^2} \cdots \frac{i}{p_2'^2 - m^2} \quad (3.56)$$

There is momentum p_i flowing in each leg. The integral over d^4y in the vertex creates a delta function that looks like $(2\pi)^4 \delta^{(4)}(p'_1 + p'_2 - p_1 - p_2)$. We also see that each propagator gives us a factor of $\frac{i}{p^2 - m^2}$.

Now when we insert this into the LSZ formula, the integral over the x 's give more delta functions that tie the momentum in each leg to the external momentum. Furthermore the LSZ formula provides a compensating factor of $p^2 - m^2$ that cancels the propagator: we find at the end for the amplitude:

$$\langle k'_1, k'_2; \text{out} | k_1, k_2; \text{in} \rangle = -(2\pi)^4 \delta^{(4)}(k'_1 + k'_2 - k_1 - k_2) i\lambda \quad (3.57)$$

I have set the factor Z in the LSZ formula to 1: it turns out there is the possibility of correcting it at $\mathcal{O}(\lambda^2)$ but we will get there in the next section. Whew. That's it. We've done it. At the end of the day, the matrix element simply goes like λ . It's really kind of a miracle: if you built a particle accelerator and smashed together these particles, this is actually the probability that the particles would scatter.

This was semi-excruciating, and from now on I will be far more schematic, but I wanted you to understand how the structure works. For future reference, let me define the *invariant matrix element* $i\mathcal{M}$ as follows:

$$\langle k_{i'}; \text{out} | k_i; \text{in} \rangle = (2\pi)^4 \delta^{(4)} \left(\sum_i k_i - \sum_{i'} k'_{i'} \right) i\mathcal{M}(k) \quad (3.58)$$

The point of this is just to strip off all the boring kinematic factors and extract the bit where the physics is. In the above example we would have

$$i\mathcal{M} = -i\lambda \quad (3.59)$$

From here, hopefully it is clear how the *momentum space Feynman rules* work:

- For each amplitude you want, draw some external legs. Put an arrow on each one to indicate which way the momentum goes: have it go inwards for in states and outwards for out states.
- Draw all possible vertices that connect together four lines. Each vertex comes with a factor $-i\lambda$.
- Put a momentum on each line. If the line is external, this is the momentum of the ingoing particle.
- For each internal line, associate a factor $\frac{i}{p^2-m^2}$. For each external line, associate a factor $\frac{i}{\sqrt{Z}}$. This is given the graphic name “amputating the propagator”.
- Make sure momentum is conserved at each vertex. This will fix many of the momenta, but not necessarily all of them – it will leave some momenta unfixed if you have loops. Integrate over any unfixed momenta. (We will see an example shortly).

This is it. You can now just draw pictures and multiply together factors, and not go through all of this excruciating work again.

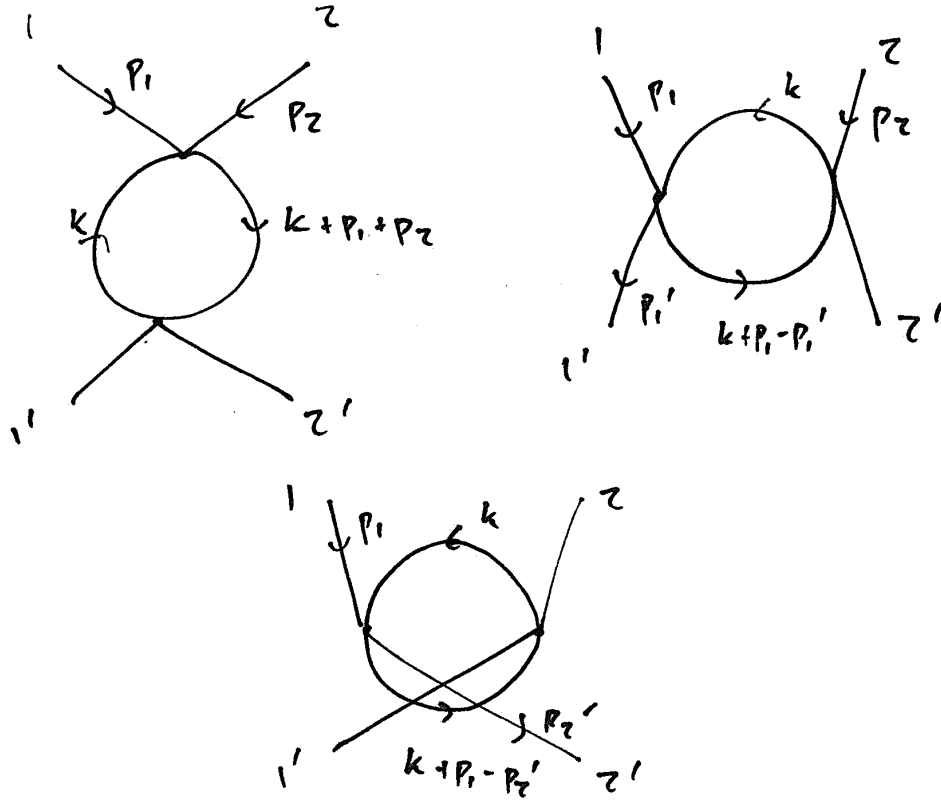


Figure 8: Loops contributing to 2-2 scattering in $\lambda\phi^4$ theory.

4 Loops and Renormalization

We now move on to one of the most fascinating and intrinsically quantum-mechanical aspects of quantum field theory: *loops*. Let's imagine calculating the scattering amplitude to the next order in λ .

4.1 Loops in $\lambda\phi^4$ theory

At order λ^2 there are three diagrams that contribute. They look like Figure 8. I may assign them as an exercise: in lecture we will *not* calculate them in detail, but let us understand what form the diagrams take. Let's remember the definition of the Mandelstam variables: if we have 2 – 2 scattering of momenta $(p_1, p_2) \rightarrow (p_1', p_2')$, then these variables are defined as

$$s = (p_1 + p_2)^2 \quad (4.1)$$

$$t = (p_1' - p_1)^2 \quad (4.2)$$

$$u = (p_2' - p_1)^2 \quad (4.3)$$

Now the first loop diagram is

$$D_1 = \frac{(-i\lambda)^2}{2} \int \frac{d^4k}{(2\pi)^4} \frac{i}{k^2 - m^2} \frac{i}{(k + p_1 + p_2)^2 - m^2} \quad (4.4)$$

Now: we are integrating over all k . Thus the integrand is a Lorentz-invariant function of the only other thing that appears in the diagram, i.e. the of the momentum $(p_1 + p_2)^2 = s$, and we may then write it as

$$D_1 = (-i\lambda)^2 iV(s) \quad (4.5)$$

What can we conclude about $V(s)$? Let us first revisit an awkward fact: the integral over k is over *all* momenta – does the integral even *converge*? At large k it is determined by the following integral:

$$I \equiv \int \frac{d^4k}{(2\pi)^4} \frac{1}{k^4} \quad (4.6)$$

If I denote the upper range of the momentum integral by Λ (where I will define precisely what I mean by this in a second), we conclude that actually this diagram does *not* converge – it instead goes like

$$D_1 \sim \lambda^2 \log(\Lambda) \quad (4.7)$$

Wow. This seems bad. Actually, it is not “bad”: like many confusing things, this divergence is secretly at the heart of all that is fun in quantum field theory. For the next few lectures, we will try to come to terms with this fact. Many discussions of this fact are a bit clouded by the fact that the expressions are actually quite complicated: thus I will make a series of simplifying assumptions to capture the relevant physics, and then you will relax these assumptions in your homework problem. First I will assume that we are scattering these particles so hard that the Mandelstam variables $s, t, u \gg m$, are far larger than m , and we may simply disregard the existence of m in these formulas.

Now let us carefully calculate the divergent part of V . This comes from calculating I . To do all such loop integrals, we always follow the following procedures:

1. **Wick-rotate.** Note the integral is over Lorentzian four-momenta. It is simpler to rotate the time coordinate of the momentum as:

$$k_0 = ik_0^E \quad (4.8)$$

in which case a Lorentzian k^2 becomes

$$(k^2)_{\text{Lorentzian}} = -((k^E)^2)_{\text{Euclidean}} \quad (4.9)$$

and the integration measure transforms as

$$\int d^4k = i \int d^4k_E \quad (4.10)$$

The integral is now

$$I = i \int \frac{d^4k_E}{(2\pi)^4} \frac{1}{(k_E)^4} \quad (4.11)$$

But now it is a spherically symmetric integral with respect to $SO(4)$ rotations in four-dimensional Euclidean space! Thus I can use the 4d Euclidean analog of polar coordinates. The key point here is that for a spherically symmetric integrand, the integration measure becomes:

$$\int d^4k_E = \text{Vol}(S^3) \int_0^\infty d|k_E| |k_E|^3 \quad (4.12)$$

where by $|k_E|$ I denote the magnitude of the Euclidean 4-vector k_E , and where $\text{Vol}(S^3)$ is the 3d surface area of a unit 3-sphere, which turns out to be $2\pi^2$. (See p193 of [1]) for more discussion of this.) Thus the integral simply becomes

$$I = \frac{i}{8\pi^2} \int_0^\infty d|k_E| \frac{1}{|k_E|} \quad (4.13)$$

2. **Regulate.** This integral is divergent. We must thus cut it off at some high momenta⁴. We will discuss the physical significance of this procedure in a moment.

There are many ways to cut off the integral; in lectures we will simply use what is called a “hard cut-off” and say the maximum Euclidean momenta that I will allow is that with $|k_E| = \Lambda$. The divergent part of the integral is then

$$I_{\text{div}} = \frac{i}{8\pi^2} \int^{\Lambda} d|k_E| \frac{1}{|k_E|} = \frac{i}{8\pi^2} \log \Lambda \quad (4.14)$$

and where this expression does *not* capture any other information. Thus, as claimed, the integral is log divergent, and now we know the precise pre-factor of this divergence. This sort of cutoff is not *gauge-invariant*, so you get into trouble if using it in gauge theories. Let’s keep that in mind for later.

Now let’s return: we have computed the divergent part of $V(s)$ as defined in (4.5). But now we note that $V(s)$ is a *dimensionless* function, and in the limit where we can neglect the mass m , the only dimensionful things entering into it are Λ and s : thus I see that $V(s) = f(\frac{\Lambda^2}{s})$. But we *know* the dependence on Λ ! Thus we see that $V(s)$ is

$$V(s) = -\frac{1}{32\pi^2} \log\left(\frac{\Lambda^2}{s}\right) + \text{const} \quad (4.15)$$

where the extra constant must be independent of s : for illustrative purposes, I will simply ignore it from now on, as we are working in the large s limit and this is the sort of thing we are neglecting. Now the other two diagrams are precisely the same with s replaced by the other Mandelstam variables t and u . We conclude that the full set of the 1-loop diagrams is this limit is:

$$D_{\text{one-loop}} = i \frac{\lambda^2}{32\pi^2} \left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right) \right] \quad (4.16)$$

Ok, now that we have calculated this, let us remember the physical interpretation. To get the full matrix element for $2 - 2$ scattering, we should add this to the previous “tree-level” bit. Thus we conclude that

$$i\mathcal{M}(s, t, u) = -i\lambda + i \frac{\lambda^2}{32\pi^2} \left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right) \right] \quad (4.17)$$

Now we need to interpret this.

4.2 Coming to terms with divergences

At first glance, this is an epic disaster! We have just calculated that the scattering amplitude depends on a parameter Λ ! This is a “cut-off”, which means that we should take it to infinity, which means that the probability for scattering together two ϕ particles is ∞ ! This is problematic. In this section I will follow the treatment of [3].

Once we finish panicking, let us discuss a few features of this computation:

1. Something interesting has happened: while trying to solve a physics problem, the problem insisted that you do an integral over *arbitrarily high momenta*. But things might happen at small scales to invalidate the calculation: note the hubris of quantum field theory: it seems to be suggesting that you need to know physics all the way down to arbitrarily small scales just to scatter together two ϕ particles? One should not need string theory to scatter two particles – surely something is wrong.

⁴This integral is also divergent for low momenta of course. If we do not omit the mass term, however, there is no issue for low momenta.

2. Relatedly, normally physics relates one observable quantity to another observable quantity. E.g. the ideal gas law:

$$PV = NkT \quad (4.18)$$

You know the temperature of a gas, and you measure how much the pressure changes when you heat it up: two observables. However, in this case, the formula above relates an observable (the scattering amplitude) to λ . Is λ an observable?

It is *not*. λ is a parameter in a Lagrangian, not something you can measure. I will thus call it the “bare coupling” for a little while. The infinity above appeared when relating an unobservable parameter in a Lagrangian to an observable: thus the infinity itself is not observable, and at the moment we are safe. Let’s try to rearrange this calculation to relate one observable quantity to another, and see what we can do.

So: what is observable? Really we are trying to give λ some kind of physical meaning. Presumably in real life if you wanted to measure λ , you would go out and scatter some particles and get some number. Let’s use this idea to *define* a new physical λ , which I will call λ_P . In other words, go to your lab and smash together some particles at a given energy (s_0, t_0, u_0) . Measure \mathcal{M} at that value, and *use* that to define λ_P :

$$i\mathcal{M}(s_0, t_0, u_0) \equiv -i\lambda_P \quad (4.19)$$

This is a *definition!* It is called a *renormalization condition*. Now from the theoretical calculation of the observable \mathcal{M} , this relates λ_P to our bare coupling λ :

$$-i\lambda_P = -i\lambda + i\frac{\lambda^2}{32\pi^2} \left[\log\left(\frac{\Lambda^2}{s_0}\right) + \log\left(\frac{\Lambda^2}{t_0}\right) + \log\left(\frac{\Lambda^2}{u_0}\right) \right] \quad (4.20)$$

Now we can solve this for λ in terms of λ_P . This is totally trivial: note that as a perturbation theory, we assume that λ_P and λ are at the same order in perturbation theory, and we only work only to quadratic order in λ and thus λ_P , which means it makes no difference which one we use in the second term.

$$\lambda = \lambda_P + \frac{\lambda_P^2}{32\pi^2} \left[\log\left(\frac{\Lambda^2}{s_0}\right) + \log\left(\frac{\Lambda^2}{t_0}\right) + \log\left(\frac{\Lambda^2}{u_0}\right) \right] + \mathcal{O}(\lambda_P^3) \quad (4.21)$$

This is totally okay to lowest order, though you may want to go home and check this for yourself. Now we use this expression to get rid of the non-physical λ in the scattering amplitude (4.17), to find:

$$\boxed{i\mathcal{M}(s, t, u) = -i\lambda_P + i\frac{\lambda_P^2}{32\pi^2} \left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right) \right]} \quad (4.22)$$

Look! The amplitude is now *finite*.

This process – what we just did – is called *renormalization*. It should really be called the far less impressive sounding “writing things in terms of observable quantities”.

Let’s discuss what happened:

1. We expressed our scattering amplitude in terms of only observable quantities; in the process of doing this, the divergence vanished all by itself. Note that we did not “add anything to make it go away”: it is simply not there in the actual observable answer. This makes sense.
2. The *relationship* between the unobservable λ and the physical λ_P involves a divergence. There is thus a sense in which the divergence is there in a parameter in the Lagrangian, but not in any observable quantity.

3. This did come at a small cost: we had to pick a point (s_0, t_0, u_0) and use that point to define our physical coupling λ_P . This is, in a sense, a loss of information: now we can only discuss physics *relative* to that point. Reflecting on this, we will need to do this for (more or less) every divergence that appears in our calculation.
4. Finally, this is a small cost, because once you specify this one bit of information, you can measure the *momentum dependence* of the scattering amplitude, which is a non-trivial function of the momenta: thus, there is plenty of predictive power in the theory.

4.3 Renormalized perturbation theory

The above is the main idea of renormalization: if you understand carefully what we did, then the conceptual part is *done*. What is not yet obvious is precisely how this might work systematically (or indeed if it *does* for all theories). In the rest of this section I will sketch out the framework to do this systematically for $\lambda\phi^4$ theory, and in the next section we will learn whether or not this is always possible.

4.3.1 Counterterms and renormalization conditions

First, I note that I am going to drop the λ_P notation that I used above. I will instead take λ to be the physical coupling; similarly I assume that m^2 is the physical mass, and I also take Z (from LSZ) to be $Z = 1$ “on-mass-shell”. These statements mean *nothing* at the moment: I need to put mathematical meaning to them with *renormalization conditions*.

However, to make the above statements work, I will instead write the Lagrangian of $\lambda\phi^4$ theory as

$$\mathcal{L} = \frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{4!}\phi^4 + \frac{1}{2}\delta_Z(\partial\phi)^2 - \frac{1}{2}\delta_m\phi^2 - \frac{\delta\lambda}{4!}\phi^4 \quad (4.23)$$

Is this a different theory from the first one? Well, not really: what I have done is things like redefine $\lambda \rightarrow \lambda + \delta\lambda$, $\phi \rightarrow \phi\sqrt{1 + \delta_Z}$, etc. A better way to think about this is that I split each term into two parts, and then I want to imagine that the second part – the “counterterm part” is something that I *adjust* order by order in perturbation theory.

This theory comes with different Feynman rules: in addition to those that we know and love, we have extra ones from the counterterms. These are shown in Figure 9.

Next, we need to establish these renormalization conditions. This is done by imagining *experiments* that fix the values of several things. First, we demand that the *exact* two-point function of the field ϕ have a pole at $p^2 = m^2$ with residue 1:

$$\int d^4x e^{ip \cdot x} \langle 0 | T \phi(x) \phi(0) | 0 \rangle = \frac{i}{p^2 - m^2} + \text{regular at } p^2 = m^2 \quad (4.24)$$

This is actually two conditions: one on the location of the pole (“weighing the particle”) and one on the residue (...I have no pithy analogy for this condition). This condition means that we are taking the Z in the LSZ formula to be

$$Z = 1 \quad (4.25)$$

This should be viewed as a choice of the normalization of the scalar field ϕ , so its often called *wave-function renormalization*.

Next, we need a renormalization condition on the coupling. We will follow what we did, and demand that the scattering amplitude (i.e. the four-point-function with the kinematic factors stripped off, as in (3.58)) at some kinematic region (s_0, t_0, u_0) is equal to λ , i.e.

$$i\mathcal{M}(s_0, t_0, u_0) = -i\lambda \quad (4.26)$$

Figure 9: Feynman rules for renormalized perturbation theory (i.e. using Lagrangian (4.23))

This condition clearly depends on the choice of (s_0, t_0, u_0) . It is customary to pick this to be, say $(s_0, t_0, u_0) = (m^2, 0, 0)$: we actually can't do this because we are assuming that all of these things are very large. I will simply keep them arbitrary to keep manifest how everything enters into the calculation.

4.3.2 Determining the counterterms

Now we use the renormalization conditions to determine what the counterterms are. Let's do the four-point interaction first. We calculate the four-point function using the new Feynman rules. To order λ^1 we find

$$i\mathcal{M}(s, t, u) = -i\lambda - i\delta_\lambda \quad (4.27)$$

Now: to satisfy the renormalization condition we conclude that

$$\delta_\lambda = 0 + \mathcal{O}(\lambda^2) \quad (4.28)$$

This is consistent with our intuition so far. Now we go to second order in λ : We now find:

$$i\mathcal{M}(s, t, u) = -i\lambda + i\frac{\lambda^2}{32\pi^2} \left[\log\left(\frac{\Lambda^2}{s}\right) + \log\left(\frac{\Lambda^2}{t}\right) + \log\left(\frac{\Lambda^2}{u}\right) \right] - i\delta_\lambda \quad (4.29)$$

(Recall this is only correct at very large s, t, u : we will keep working in this limit for pedagogical simplicity). Now we have something to do. We now impose the renormalization condition (4.26), which fixes the counterterm, telling us that

$$\boxed{i\delta_\lambda = i\frac{\lambda^2}{32\pi^2} \left[\log\left(\frac{\Lambda^2}{s_0}\right) + \log\left(\frac{\Lambda^2}{t_0}\right) + \log\left(\frac{\Lambda^2}{u_0}\right) \right]} \quad (4.30)$$

This is the point of the analysis. Note that if we now use these together the amplitude becomes:

$$i\mathcal{M}(s, t, u) = -i\lambda + i\frac{\lambda^2}{32\pi^2} \left[\log\left(\frac{s_0}{s}\right) + \log\left(\frac{t_0}{t}\right) + \log\left(\frac{u_0}{u}\right) \right] \quad (4.31)$$

$$\begin{aligned}
 \text{---} \bigcirc \text{---} &= \frac{i}{p^2 - m^2} + \text{REGULAR AT } p^2 = m^2 \\
 \begin{array}{c} \diagup \\ \bigcirc \\ \diagdown \\ \diagup \\ \diagdown \end{array} &= -i\lambda \\
 (s, t, u) &= (s_0, t_0, u_0)
 \end{aligned}$$

Figure 10: The renormalization conditions

$$\begin{array}{ccccccccc}
 \diagdown & + & \diagup & + & \diagdown & + & \diagup & + & \diagdown \\
 \diagup & & \diagdown & & \diagup & & \diagdown & & \diagup \\
 \bullet & & \bigcirc & & \bigcirc & & \bigcirc & & \bigcirc \\
 -i\lambda & & \lambda^2(\dots) & & & & & & -i\delta\lambda
 \end{array}$$

Figure 11: Diagrams contributing to renormalization of coupling λ :

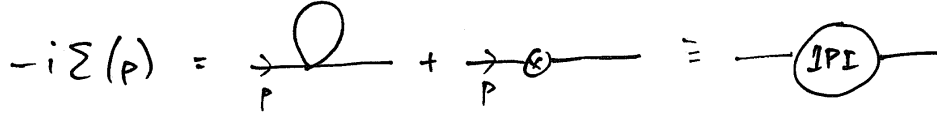


Figure 12: Diagrams contributing to renormalization of propagator

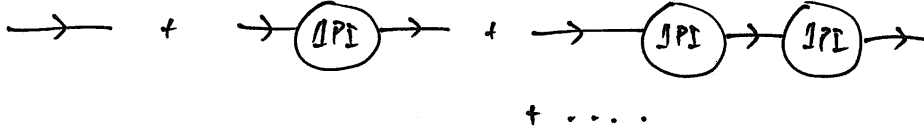


Figure 13: Infinite geometric sum contributing to corrected propagator.

i.e. it is manifestly finite in terms of the renormalized (“physical”) coupling. This is of course the same expression I wrote earlier (with the λ_P), but this way of doing it should make it more obvious how to go to higher order in perturbation theory: δ_λ will be corrected order by order, but it will always work.

Now we should do the same for the two-point function. This actually involves an interesting wrinkle: we seek to keep track of how the *pole* shifts in perturbation theory. To do this, we need to sum an infinite set of Feynman diagrams. Let’s try to understand the structure of corrections to the two point function. To order λ^0 , we have the free result:

$$\int d^4x e^{ip \cdot x} \langle 0|T\phi(x)\phi(0)|0\rangle = D_F(p) \quad (4.32)$$

To order λ^1 we have the following corrections, whose amputated form I denote by $-i\Sigma(p)$ (factor of $-i$ for later convenience)

At this point I should mention that there is a better definition of $-i\Sigma(p)$: to extend the definition of $-i\Sigma(p)$ to all orders in perturbation theory, one should define it to be $-i\Sigma(p)$ to be the sum of all 1PI diagrams, where 1PI stands for “one-particle-irreducible”, and where a diagram is 1PI if you cannot break it into two by cutting a single line. See Problem Sheet 2 for further development of this.

Here we do *not* amputate the propagators (as we’re not doing LSZ), so the sum of these two terms is $D_F(p) + D_F(p)(-i\Sigma(p))D_F(p)$. However we can also now consider summing up all the contributions, as shown in Figure 13:

This is an infinite sum! Luckily it is a geometric sum:

$$D_F(p) \sum_{n=0}^{\infty} (-i\Sigma(p)D_F(p))^n = D_F(p) \frac{1}{1 + i\Sigma(p)D_F(p)} = \boxed{\frac{i}{p^2 - m^2 - \Sigma(p)}} \quad (4.33)$$

Thus we see that actually $\Sigma(p)$ (evaluated at $p^2 = m^2$) is the shift in the mass. Similarly, it is possible to show that $\frac{d\Sigma(p)}{dp^2} \Big|_{p^2=m^2}$ is the shift in the residue (see Problem Sheet 2).

Thus in terms of $\Sigma(p)$, the renormalization conditions (4.24) that we want to satisfy are:

$$\Sigma(p^2 = m^2) = 0 \quad \frac{d\Sigma(p)}{dp^2} \Big|_{p^2=m^2} = 0 \quad (4.34)$$

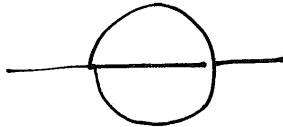


Figure 14: “Sunset” diagram that contributes to δ_Z at $\mathcal{O}(\lambda^2)$.

Now, let’s evaluate the diagram. We find

$$-i\Sigma(p^2) = -i\lambda \frac{1}{2} \int d^4k \frac{i}{k^2 - m^2} + i(p^2\delta_Z - \delta_m) \quad (4.35)$$

This loop part of the diagram doesn’t depend on p^2 (the momentum in!). This this makes it quite easy to satisfy the renormalization condition: we have

$$\delta_Z = 0 \quad \delta_m = -\lambda \frac{1}{2} \int d^4k \frac{i}{k^2 - m^2} \quad (4.36)$$

I won’t carefully evaluate δ_m , as it depends crucially on which form of cutoff we use. If we use a hard cutoff Λ like before, we find something like

$$\delta_m \sim \lambda\Lambda^2 \quad (4.37)$$

With these choices, we see that we have

$$\Sigma(p^2) = 0 \quad (4.38)$$

to first order in λ . This is the analog of (4.31); it’s just a coincidence that it became exactly zero, if we go to higher orders in perturbation theory like Figure 14. These diagrams give us a non-trivial dependence on the ingoing momentum p that becomes finite after renormalization. Take a look at problem 10.3 on p345 of Peskin if you want to see how this works in practice.

As you can see, in this example renormalization is simply bookkeeping. Is this always the case?

4.4 What theories are renormalizable and what does this mean?

We now turn to a systematic study of which theories are renormalizable. Roughly speaking, what happened above was that for every possible divergence we added a counterterm. For every counterterm we needed a renormalization condition; thus if we have only *finitely* many divergences we will need only finitely many renormalization conditions and everything is okay: we say the theory is *renormalizable*. However if we have infinitely many divergences, then we need infinitely many renormalization conditions, and then the theory will end up with no predictive power: we would say it is *non-renormalizable*.

Let us now try to figure out a rule for what diagrams are divergent. Let’s generalize away from the $\lambda\phi^4$ in 4 dimensions and consider instead the ϕ^n theory in d dimensions:

$$S[\phi] = \int d^d x \left(\frac{1}{2}(\partial\phi)^2 - \frac{1}{2}m^2\phi^2 - \frac{\lambda}{n!}\phi^n \right) \quad (4.39)$$

Now: we are first going to do some dimensional analysis. Note that we normally have

$$E = \hbar\omega \quad x = ct \quad E = mc^2 \quad (4.40)$$

etc. But as we have set $\hbar = c = 1$, energies have the same units as mass which have the same units of inverse length. Thus any quantity has dimensions of mass raised to some power: e.g. an energy has units of $[\text{mass}]^1$ and a length has units of $[\text{mass}]^{-1}$. From now on the “dimension” of a quantity denotes this power: e.g. I might say “length has dimension -1 ”. Square brackets around a quantity denote its dimension.

Now the action must be dimensionless, as it appears in the exponential e^{iS} in the path integral. As $S = \int d^d x \mathcal{L}$, this means that the Lagrangian density has dimension d . We may thus figure out the dimension of the field itself and the coupling λ :

$$[\mathcal{L}] = d \quad [\phi] = \frac{d-2}{2} \quad [\lambda] = d - n \left(\frac{d-2}{2} \right) \quad (4.41)$$

Now let me define the “superficial degree of divergence” D of a diagram: the superficial degree of divergence is how badly a diagram *looks* like it diverges, i.e. if the cutoff is Λ then the diagram looks like Λ^D .

If a diagram has L loops and P propagators, then every propagator contributes a p^2 in the denominator and every loop contributes a $d^d p$ in the numerator: thus we have

$$D = dL - 2P \quad (4.42)$$

The more loops, the worse the divergence: the more propagators, the better. In practice, the actual degree of divergence may differ from the superficial degree: we will return to these technicalities shortly, but for now we assume that this definition of D is a good handle on the actual divergences.

First we will derive an expression for D from dimensional analysis. Consider a diagram S with N external legs and V vertices. What is the dimension $[S]$ of this diagram? The dimension actually only depends on N : one way to understand this is to *imagine* that we are considering a field theory with an interaction term $\eta \phi^N$ in it. (Note we are not adding this term to our theory: it is just a fictitious device for dimension-counting). The diagram might then receive a contribution that goes like η (with no extra dimensionful factors), and thus we must have $[S] = [\eta]$. But from the dimension counting in (4.41), we conclude that

$$[\eta] = d - N \left(\frac{d-2}{2} \right) \quad (4.43)$$

However, another way to construct the diagram is by sewing together many propagators and vertices etc. in the actual theory. If there are V vertices, then the most divergent part of the diagram must go (by definition) as $\lambda^V \Lambda^D$; note that any other dimensionful scales (e.g. external momenta, etc.) will only decrease the degree of divergence. By equating the two ways of computing the dimension, we conclude that

$$d - N \left(\frac{d-2}{2} \right) = [\lambda^V \Lambda^D] = V[\lambda] + D = V \left(d - n \left(\frac{d-2}{2} \right) \right) + D \quad (4.44)$$

We solve this for D to find

$$\boxed{D = d + V \left(n \left(\frac{d-2}{2} \right) - d \right) - N \left(\frac{d-2}{2} \right)} \quad (4.45)$$

The last term is always negative or 0 for all reasonable dimensions⁵. Now we can understand that the physics depends crucially on the sign of the thing multiplying V , which we note is simply the (inverse) mass dimension of λ . There are three possibilities:

⁵What, you ask, of the case $d = 1$, i.e. quantum mechanics? It may appear that the term proportional to N has a different sign, suggesting that divergences are especially bad there. This makes little sense: and indeed (**extra credit homework problem**) if one takes into account how the number of vertices V is constrained for a given amplitude one does find that quantum mechanics is power-counting renormalizable, as one might expect. (The consistency of quantum-mechanics with these formulas was explained to me by a member of this class.)

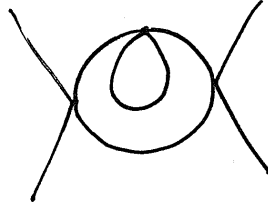


Figure 15: This contains a quadratically divergent subdiagram though its superficial degree of divergence D is 0.

1. $[\lambda] > 0$, or **Super-renormalizable**: in this case, increasing V makes every diagram less divergent: thus there are only finitely many divergent diagrams, period. These theories can then definitely be renormalized for with a finite number of counterterms, and new divergences will stop happening at sufficiently high order in perturbation theory (i.e. in V). This tends to happen in low dimension: e.g. ϕ^4 theory in $d = 3$ falls into this class.
2. $[\lambda] = 0$, or **Renormalizable**: in this case, the bit proportional to V vanishes. Now there are still only finitely many divergent *amplitudes*, as for sufficiently large N , D is still 0. Nevertheless, we will find more and more divergences as we go to higher orders in perturbation theory. We thus need to continue adjusting the counterterms to arbitrarily high order.
3. $[\lambda] < 0$, or **Non-renormalizable**: in this case, we cannot renormalize: as we increase V , we have a larger and larger window of N where we can have a $D > 0$. Thus there are infinitely many divergences, and we cannot absorb them into finitely many counterterms.

This closes the classification of field theories; what we just did is called “power-counting renormalization”. The key thing here is that everything is controlled by the *mass dimension* of the coupling: if ever it has negative mass dimension, then the theory will be non-renormalizable. For example, if we just add a $\eta\phi^6$ term to our beloved ϕ^4 theory in four dimensions, it will not be renormalizable any more. I will discuss the physical significance of this in the next section.

There is a last worry: does the superficial degree of divergence adequately describe the degree of divergence of the diagram? It actually does *not*, for two distinct reasons:

1. It might overstate the divergence: this could happen if some more sophisticated effect (e.g. a symmetry) sets the diagram to zero or reduces the divergence level. This happens in QED, for example.
2. It might understate the divergence: this happens if there is a badly divergent *subdiagram*, e.g. as in Figure 15, which from the formula has $D = 0$ but actually has a clearly quadratically divergent bit. This turns out to not upset the reasoning above, because it is not a “new” divergence: the counterterm does its magic in the subdiagram.

Despite these subtleties, for most purposes the conclusions of “power-counting renormalization” give a good physical picture as to what is happening in the actual theory, though proving it can be difficult.

4.5 A few non-renormalizable theories

Here we will try to understand what it really means to be non-renormalizable. First, let’s try to understand what the dimension of the coupling *means*. More discussion of this in language very similar to that which I

use here can be found in Chapter III.2 of [3]. Let's consider adding a non-renormalizable term like ϕ^6 to our usual 4d action. In 4d, $[\phi] = 1$ and so we have:

$$\mathcal{L}_6 = \int d^4x \frac{c}{M^2} \phi^6 \quad (4.46)$$

where M has dimensions of mass and c is dimensionless. The claim is that adding this term makes the theory non-renormalizable, which means that we can no longer take the cutoff to infinity and retain predictive power. Why is this? A clue is given by the following question: let's imagine computing some sort of an observable – say a scattering amplitude at energy E – and asking how it depends on E to first order in c . Scattering amplitudes are (at the end of the day), probabilities, and are thus dimensionless. Thus we see that the contribution from the ϕ^6 term must, from dimensional analysis, take the form

$$i\mathcal{M}_6 \sim \frac{c}{M^2} E^2 \quad (4.47)$$

The E^2 must appear in the numerator to soak up the factor of M^2 from the denominator. Thus, no matter what, this contribution is unimportant at low-energies: but we also see that no matter what, eventually when $E \sim M$ perturbation theory in c will break down. Thus this theory seems to become *strongly coupled in the UV*: crudely speaking, this is what non-renormalizability hints at, it says that we *do not know about what is happening in the UV*: this is why we cannot extend the cutoff to infinity.

So what should one do, then, if one is handed a non-renormalizable theory? Since you cannot take the cutoff Λ to infinity, you should take the idea of the cutoff seriously: have it hanging around your calculation, and just try to ask questions at energies $E \ll \Lambda$: hopefully then the cutoff will not affect your life too much. This is the idea behind *effective field theory*, and will be more correctly developed in C. Sleight's course on renormalization next term, which I encourage everyone to take.

In the meantime, however, I will introduce two of my favorite non-renormalizable theories to explain how these ideas affect low-energy physics:

4.5.1 Pions

Pions are real-life particles that you have probably heard of: they have a mass of roughly 135 MeV, and there are three kinds of them. We can represent them by a triplet of real scalar fields $\vec{\pi}(x)$. In a particular limit (massless quarks), we can choose to ignore their mass and their action looks like

$$S = \int d^4x \left(\frac{1}{2} (\partial_\mu \vec{\pi} \cdot \partial^\mu \vec{\pi}) + \frac{1}{2F_\pi^2} \partial_\mu \vec{\pi} \cdot \pi \partial_\mu \vec{\pi} \cdot \vec{\pi} + \dots \right) \quad (4.48)$$

From here we see that F_π has dimensions of mass: it is called the pion decay constant, and in real life it is 93 MeV. Note that this coupling thus has negative mass dimension, and so the theory is non-renormalizable. You can, however, use this theory at low energies: e.g. if you try to scatter together pions at energy E then from here their scattering amplitude will go like

$$i\mathcal{M} \sim \frac{E^2}{F_\pi^2} \quad (4.49)$$

This is okay at low energies, but perturbation theory seems to break down at $E \sim F_\pi$, and thus from the pion Lagrangian alone we do not know what to do.

I note, however, that in real life, we *do* actually know what happens: pions are actually made of quarks that are held together by the strong interactions. At high energies, they come loose: this “coming loose” is what made the pion theory break down, and then we need to use the full theory of the strong interactions, which is called QCD and *is* renormalizable.

4.5.2 Gravity

Everyone knows about gravity. From S. Ross's course you even know the gravitational action. It is

$$S = \frac{1}{16\pi G_N} \int d^4x \sqrt{-g} R \quad (4.50)$$

Let's do something bold and reckless: let's *treat gravity as a quantum field theory*. We will do this by expanding around flat space, i.e. write

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (4.51)$$

and then treating $h_{\mu\nu}$ as a quantum field. If we plug this ansatz into the action above, we find schematically that $R \sim (\partial h)^2$, and so we get (very schematically) something like:

$$S = \frac{1}{16\pi G_N} \int d^4x (\partial h)^2 (1 + h^2 + h^4 + \dots) \quad (4.52)$$

where there are actually arbitrarily many terms in h (coming, e.g., from inverting the metric, the square-root of the determinant, etc.) and I am suppressing all indices as they are really quite bad.

Note from (4.51) that h is dimensionless, and thus G_N must have mass dimension of -2 : the resulting dimensionful scale is called the *Planck scale*, and is 10^{19} GeV.

$$\frac{1}{16\pi G_N} \equiv m_{Pl}^2 \quad (4.53)$$

To make this look more like a normal quantum field theory, let's rescale:

$$\hat{h} = m_{Pl} h \quad (4.54)$$

and we now find

$$S = \int d^4x (\partial \hat{h})^2 \left(1 + \frac{1}{m_{Pl}^2} \hat{h}^2 + \frac{1}{m_{Pl}^4} \hat{h}^4 + \dots \right) \quad (4.55)$$

There is a coupling constant with negative mass dimension in the Lagrangian. Thus, gravity – viewed as a quantum field theory – is non-renormalizable. You have probably heard that there are some difficulties with quantizing gravity: this is what people mean. Having said that, everything is fine at low energies: it is only at high energies (i.e. around the Planck scale) that things get difficult. Unlike in the pion case, here we do *not* know what to do, due to a pesky lack of experiments, but string theory is a candidate for a quantum theory of gravity works perfectly fine at high energies. A nice review of the effective field theory of gravity is [6].

5 Global symmetries in the functional formalism

Here we will briefly discuss how global symmetries work in the path integral. Recall that an example of a theory with a global symmetry is the complex scalar field, whose action is

$$S[\phi, \phi^\dagger] = \int d^4x ((\partial\phi)^\dagger(\partial\phi) - m^2\phi^\dagger\phi) \quad (5.1)$$

which is clearly invariant under phase rotations of the scalar field

$$\phi \rightarrow e^{i\alpha\phi} \quad \delta_\phi = i\alpha\phi \quad (5.2)$$

where the second equality is the infinitesimal action of the symmetry operation.

In this lecture, we will be instead more general, though maybe it is helpful to keep the complex scalar field in mind: consider a field theory with a bunch of fields ϕ^a with an action that is invariant under a general global symmetry, i.e. under

$$\phi^a(x) \rightarrow \phi^a(x) + \epsilon\delta\phi^a(x) \quad (5.3)$$

that leaves the action invariant:

$$S[\phi^a + \epsilon\delta\phi^a] = S[\phi^a] \quad (5.4)$$

5.1 Classical Noether's theorem

As you are well aware, such global symmetries lead to conserved currents via Noether's theorem. Let's review the classical version of this theorem (possibly in a slightly different form than you are used to). First, we perform a trick: let us consider the variation (5.3) with a symmetry parameter that *depends on x*:

$$\phi^a(x) \rightarrow \phi^a(x) + \epsilon(x)\delta\phi^a(x) \quad (5.5)$$

(Note: no matter what it looks like, this has nothing to do with gauge symmetry, as we have not yet discussed gauge symmetry. This is just a trick). Now the action is *not* invariant under this transformation; however it *is* invariant if the parameter ϵ is constant, which means that its variation must be proportional to a derivative. By locality of the action⁶, this means:

$$\delta_\epsilon S[\phi^a] = \int d^4x j^\mu(x)\partial_\mu\epsilon(x) = - \int d^4x \partial_\mu j^\mu(x)\epsilon(x) \quad (5.6)$$

where I have *defined* j^μ to be the “thing that multiplies $\partial_\mu\epsilon$ ”, and in the second equality I have integrated by parts. This is, in general, not zero (why would it be?).

However, let us now imagine that the field ϕ^a obeys the equations of motion, i.e. that we are “on-shell”. In that case the variation of the action under *all* variations of the field ϕ is zero:

$$\delta_\phi S[\phi^a] = 0 \quad (5.7)$$

and in *particular* under the specific variation (5.4). This means that the right hand-side of (5.6) is zero, and thus that

$$\boxed{\partial_\mu j^\mu(x) = 0 \quad \text{on-shell}} \quad (5.8)$$

In other words, we have shown that we *have a current* j^μ , and that the current is conserved *classically*, when the fields obey the classical equations of motion. We used both the invariance of the action *and* the fact that we were on-shell. This also gives you a way to compute j^μ , and it is instructive to verify that it gives you what you expect in the usual case.

⁶i.e. using the fact that the action is an integral of a local Lagrangian density.

5.2 Quantum Ward identities

Now let us study this phenomenon in the quantum theory, i.e. we study the path integral

$$Z = \int [\mathcal{D}\phi^a] \exp(iS[\phi^a]) \quad (5.9)$$

and perform a similar operation. We consider the symmetry transformation given by (5.4) as defining a *change of variables in the path integral*:

$$\phi^a(x) \rightarrow \phi'^a(x) = \phi^a(x) + \epsilon(x)\delta\phi^a(x) \quad (5.10)$$

The first thing to note is that this is a change of variables, and thus does not alter the numerical value of the functional integral:

$$\int [\mathcal{D}\phi^a] \exp(iS[\phi^a]) = \int [\mathcal{D}\phi'^a] \exp(iS[\phi'^a]) \quad (5.11)$$

This is always true for any change of variables, whether a symmetry operation or not. Now, because this is a symmetry operation, you can convince yourself by discretizing the measure that the measure of the path integral (usually) does not change⁷, and we have

$$[\mathcal{D}\phi^a] = [\mathcal{D}\phi'^a] \quad (5.12)$$

The action however does change a little bit: we see that we have (to first order in $\epsilon(x)$)

$$S[\phi'^a] = S[\phi^a] - \int d^4x \partial_\mu j^\mu(x) \epsilon(x) \quad (5.13)$$

Putting all these things on the right-hand side, we see that we have:

$$\int [\mathcal{D}\phi^a] \exp(iS[\phi^a]) = \int [\mathcal{D}\phi^a] \left(1 - i \int d^4x \partial_\mu j^\mu(x) \epsilon(x) + \mathcal{O}(\epsilon^2) \right) \exp(iS[\phi^a]) \quad (5.14)$$

The 1 bit cancels: but the bit involving $\partial_\mu j^\mu$ is precisely what we use to calculate the expectation value of $\partial_\mu j^\mu$ from the path integral! As this must hold for any $\epsilon(x)$, we conclude that we *must* have:

$$\langle 0 | \partial_\mu j^\mu(x) | 0 \rangle = 0 \quad (5.15)$$

In other words, the expectation value of the current in the vacuum is zero! This is *part of* the quantum version of Noether's theorem. From this, and from our classical intuition might be tempted to conclude that $\partial_\mu j^\mu = 0$ as an *operator*, i.e. it will vanish in all states. This is actually not quite true. To understand this, let's apply this not to the generating functional itself, but instead to the expectation value of

$$\langle 0 | T \phi^b(x_1) \phi^c(x_2) | 0 \rangle = \frac{1}{Z[0]} \int [\mathcal{D}\phi^a] \phi^b(x_1) \phi^c(x_2) \exp(iS[\phi^a]) \quad (5.16)$$

But by the arguments above, this quantity is also numerically equal to

$$\frac{1}{Z[0]} \int [\mathcal{D}\phi'^a] \phi'^b(x_1) \phi'^c(x_2) \exp(iS[\phi'^a]) \quad (5.17)$$

Where we have followed the time-honored practice of “putting primes everywhere”. Now we connect everything to the unprimed variables using the symmetry transformation. We find

$$\frac{1}{Z[0]} \int [\mathcal{D}\phi^a] (\phi^b(x_1) + \epsilon(x_1)\delta\phi^b(x_1)) (\phi^c(x_2) + \epsilon(x_2)\delta\phi^c(x_2)) \left(1 - i \int d^4x \partial_\mu j^\mu(x) \epsilon(x) + \mathcal{O}(\epsilon^2) \right) \exp(iS[\phi^a]) \quad (5.18)$$

⁷A somewhat surprisingly beautiful exception to this fact arises in the case of *anomalies*, which happen precisely when this statement is wrong and the measure does transform under the symmetry. It turns out that the possible ways it can transform are very restricted. Look at Chapter 19 of Peskin.

Setting this equal to its unprimed version, we see that everything multiplying an ϵ must vanish. Those terms are

$$\epsilon(x_1)\delta\phi^b(x_1)\phi^c(x_2) + \epsilon(x_2)\phi^b(x_1)\delta\phi^c(x_2) - i\phi^b(x_1)\phi^c(x_2) \int d^4x \epsilon(x)\partial_\mu j^\mu(x) \quad (5.19)$$

where it is all multiplied by $\int [\mathcal{D}\phi^a] \exp(iS[\phi^a])$, and thus we are computing the vev of these quantities. Taking $\epsilon(x)$ to be $\delta(z) = \delta^{(4)}(y - z)$ function have just shown that

$$\boxed{i\langle 0|\phi^b(x_1)\phi^c(x_2)\partial_\mu j^\mu(y)|0\rangle = \langle 0|\delta\phi^b(x_1)\phi^c(x_2)|0\rangle\delta^{(4)}(y - x_1) + \langle 0|\phi^b(x_1)\delta\phi^c(x_2)|0\rangle\delta^{(4)}(y - x_2)} \quad (5.20)$$

(where all expectation values are time-ordered). Compare this to the case when $\partial_\mu j^\mu$ was by itself, in (5.15): we see that it is no longer *quite* conserved. In other words, if we look at correlation functions of $\partial_\mu j^\mu$ with other operators that are charged under the symmetry associated with j^μ , then it vanishes up to delta function terms where those operators are inserted. It should be easy to understand how this works if you insert more operators.

This is called the *Ward identity* associated with the symmetry: it is the quantum manifestation of the classical Noether's theorem. I believe you have encountered similar things in QED, associated with the conservation of electric charge.

Our discussion has been very abstract, so I leave to you to work out the details of how this works for a specific symmetry (e.g. the $U(1)$ phase rotations of the complex scalar in (5.1)).

6 Fermions

We will now finally move beyond scalar fields. In this section we will discuss fermions from the point of view of the path integral – to be more precise, we will only discuss *Dirac* fermions. Let us first review a few aspects of Dirac fermions from the canonical point of view.

6.1 Fermions in canonical quantization

Recall from IFT that there is such a thing as a Dirac field $\psi(x)$, whose canonical commutation results in *fermions*. More precisely, one can write down the Dirac Lagrangian:

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi} (i\rlap{\not{D}} - m) \psi, \quad (6.1)$$

the variation of which gives you the celebrated Dirac equation:

$$(i\rlap{\not{D}} - m) \psi = 0 \quad \rlap{\not{D}} \equiv \gamma^\mu \partial_\mu \quad (6.2)$$

In IFT ψ was then promoted to an operator that acted on the Hilbert space.

There are two things that are *different* about the Dirac field relative to the scalar field that we have studied:

1. ψ is actually a four-component object called a *spinor*, which means that under a Lorentz transformation $\Lambda \in SO(3, 1)$, we have an expression that looks like this:

$$U(\Lambda)^\dagger \psi_a(x) U(\Lambda) = M(\Lambda)^a_b \psi_b(x) \quad (6.3)$$

where a, b run over 4 spinor indices, $U(\Lambda)$ is the operator that represents the Lorentz transformation on the QFT Hilbert space, and $M(\Lambda)$ is the matrix that acts on spinor indices that is in the spin- $\frac{1}{2}$ representation of the Lorentz group (which one can construct out of gamma matrices, see e.g. Chapter 3 of [1] if this is not familiar). Thus the Dirac field has spin $\frac{1}{2}$.

2. The Dirac field obeys *anti-commutation* relations rather than commutation relations. In other words, the analog of the canonical commutator for the scalar field ϕ is the following canonical anticommutator:

$$\boxed{\{\psi_a(\vec{x}), \psi_b^\dagger(\vec{y})\} = \delta^{(3)}(\vec{x} - \vec{y}) \delta_{ab}} \quad \{\psi_a(\vec{x}), \psi_b(\vec{y})\} = \{\psi_a^\dagger(\vec{x}), \psi_b^\dagger(\vec{y})\} = 0 \quad (6.4)$$

This is an important fact: it means that fermions obey Fermi-Dirac statistics. In usual quantum mechanics this means that the wavefunction is antisymmetric as a function of fermion coordinates. In QFT this information is encoded in the following way: if we write the Dirac field in terms of creation and annihilation operators:

$$\psi(x) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} \sum_s \left(a_{\vec{p}}^s u^s(\vec{p}) e^{-ip \cdot x} + b_{\vec{p}}^{s\dagger} v^s(\vec{p}) e^{+ip \cdot x} \right) \quad (6.5)$$

Then we find that the a and b 's obey these relations:

$$\{a_{\vec{p}}^r, a_{\vec{q}}^s\} = \{b_{\vec{p}}^r, b_{\vec{q}}^s\} = (2\pi)^3 (2E_{\vec{p}}) \delta^{(3)}(\vec{p} - \vec{q}) \delta^{rs} \quad (6.6)$$

with all other anticommutators 0. In particular, we have

$$\{a_{\vec{p}}^{r\dagger}, a_{\vec{p}}^{r\dagger}\} = 2(a_{\vec{p}}^{r\dagger})^2 = 0 \quad (6.7)$$

which means that we can never put two fermions with the same momentum \vec{p} and spin r in the same state! (This is important, as it is the only reason that all the electrons in all the atoms in the periodic table do not all simply collapse into the lowest 1S orbital. This would probably make chemistry easier but I feel it would be bad for the existence of complex structures).

It turns out that these two facts that are *a priori* unrelated – the fact that the Dirac field has spin $\frac{1}{2}$ and the fact that it obeys anticommutation statistics – are actually related by the *spin-statistics theorem*, which tells you that to have a Lorentz-invariant and unitary theory you must quantize spin $\frac{1}{2}$ fields with anticommutation relations. It is instructive to go through quantizing it the wrong way to see what happens, as I imagine you did in IFT.

One final thing that I discuss from the canonical formalism is the form of the Hamiltonian, which turns out to be

$$H = \sum_s \int \frac{d^3p}{(2\pi)^3 2E_{\vec{p}}} E_{\vec{p}} \left(a_{\vec{p}}^{s\dagger} a_{\vec{p}}^s + b_{\vec{p}}^{s\dagger} b_{\vec{p}}^s - 2 \left(\frac{1}{2} \right) \right) \quad (6.8)$$

The thing of interest here is the vacuum energy bit, i.e. the last term: note that it appears with a *minus sign*.

6.2 Grassman variables

Now that we have reviewed everything that we know about fermions, we now turn to what we did for scalars in the first half of this course: we seek to get the fermion physics from the path integral. How can we do this? It should be clear that we will need some new ingredients, as we previously did:

$$\text{Integral over ordinary numbers} = \text{Commuting fields in quantum theory} \quad (6.9)$$

Now we need

$$\text{Integral over something} = \text{Anti-commuting fields in quantum theory} \quad (6.10)$$

The question is: what is the *something*? It turns out that we will need to develop a *new sort of number* called a Grassman variable. My treatment follows Section 9.5 of [1] and Chapter 44 of [4]; in particular, I refer you to the latter for a more detailed discussion.

6.2.1 Anticommuting numbers

As in the quantum theory the resulting operators should anticommute, it seems reasonable to demand that we want a sort of ordinary *number* that also anticommutes. So let us just start defining things and see how far we can go. I define a Grassman number as an objects η, θ that anticommute, so:

$$\theta\eta = -\eta\theta \quad (6.11)$$

This means that the square of any Grassman number vanishes:

$$\theta\theta = -\theta\theta = 0 \quad (6.12)$$

This fact makes algebra refreshingly easy. We can multiply any Grassman numbers by normal numbers a, b in the usual way, with all the normal rules. Note that the product of two Grassman numbers $\theta\eta$ commutes with any *other* Grassman numbers: thus, for purposes of manipulation we can imagine $\theta\eta$ to be a normal number.

Let's now consider a function of a single Grassman numbers $f(\theta)$. We can always Taylor expand in powers of θ : the most general function is

$$f(\theta) = A + B\theta \quad (6.13)$$

where all the other possible terms in the Taylor expansion are *zero* due to Grassman-ness. This is nice and simple. A brief aside: we are typically going to be interested in commuting functions of anticommuting numbers: if this is the case for f , then we see that A is commuting and B is anticommuting.

Let's now do calculus: how do we take a derivative with respect to a Grassman number? There is a little bit of ambiguity in whether we take the derivative from the left or the right. From the right we *define*

$$f(\theta)\overleftarrow{\partial}_\theta = B \quad (6.14)$$

and thus from the *left* we thus have

$$\partial_\theta f(\theta) = -B, \quad (6.15)$$

where we anticommuted B and θ before taking the derivative. Great, derivatives were easy. Now let's do integrals! Our ultimate goal is to do path integrals, so we will need to perform the analogue of $\int_{-\infty}^{+\infty} dx$ for an ordinary integral. So what is

$$\int d\theta f(\theta) = \int d\theta(A + B\theta) \quad (6.16)$$

The integral should be linear in f , so it should be a linear function of A and B . Its value can be fixed by appealing to another property: in our derivation of Ward identities, Gaussian integrals, etc. we made heavy use of shifting the integration variable, so it would be nice if we could define Grassman integration to be invariant under this property. In other words, we want

$$\int d\theta f(\theta) \equiv \int d\theta f(\theta + \eta) = \int d\theta(A + B\eta + B\theta) \quad (6.17)$$

As a function of θ , the shift changes the constant term, but leaves the linear term in θ unchanged. Thus the only possible integral that is shift-invariant is a constant times the coefficient of the linear term. For convenience we take this constant to be 1, and so we *define*

$$\boxed{\int d\theta(A + B\theta) \equiv B} \quad (6.18)$$

Above I quietly assumed that B was commuting as I didn't worry about the ordering of B, θ .

6.2.2 Integrating many anticommuting numbers

If we instead integrate multiple Grassman variables there is a sign ambiguity, and we will thus use the convention

$$\int d\theta \int d\eta(\eta\theta) = +1 \quad (6.19)$$

(i.e. we do the innermost integral first).

We now move to having *multiple* Grassman variables θ_i , where $i \in 1 \dots n$. We now want to understand how changes of variables work in integration. Consider having θ_i and an invertible matrix U_{ij} such that $\theta'_i = U_{ij}\theta_j$. Then consider the product:

$$\prod_i \theta'_i = \frac{1}{n!} \epsilon^{i_1 i_2 \dots i_n} \theta'_{i_1} \theta'_{i_2} \dots \theta'_{i_n} \quad (6.20)$$

$$= \frac{1}{n!} \epsilon^{i_1 i_2 \dots i_n} U_{i_1 j_1} \theta_{j_1} U_{i_2 j_2} \theta_{j_2} \dots U_{i_n j_n} \theta_{j_n} \quad (6.21)$$

$$= \frac{1}{n!} \epsilon^{i_1 i_2 \dots i_n} U_{i_1 j_1} U_{i_2 j_2} \dots U_{i_n j_n} \epsilon^{j_1 j_2 \dots j_n} \prod_j \theta_j \quad (6.22)$$

$$= (\det U) \prod_i \theta_i \quad (6.23)$$

In other words, the product of many θ 's transforms as a power of the determinant. Let us now understand what this means for integration: consider a general integral

$$\int d^n \theta'_i f(\theta_i) \quad (6.24)$$

where we defined

$$d^n \theta_i = \prod_i d\theta_i \quad (6.25)$$

The only term that can possibly survive this integral is one which has a product of all possible θ_i , i.e. the term proportional to a in the expansion of $f(\theta_i) = \dots + a \prod_i \theta_i + \dots$. We can now write this as a product over the primed θ_i 's using the expression above. We can then simply evaluate the integral; however the factor of $\det U^{-1}$ that we obtain remains, leaving us with the expression:

$$\int d^n \theta' f(\theta_i) = \int d^n \theta' \prod_i \theta_i = (\det U)^{-1} \int d^n \theta' f(\theta'_i) = (\det U)^{-1} \int d^n \theta f(\theta_i) \quad (6.26)$$

In the last step we have simply relabelled θ'_i to θ_i . One way to interpret this is that the *measure* for Grassman integration transforms in the opposite way than we are used to for normal commuting integration (note the factor of $\det U$ would have been in the numerator then).

As the Dirac field is complex, we will also need complex Grassman numbers θ, θ^* . We define complex conjugation to reverse the order of products, as for Hermitian conjugation of operators

$$(\theta\eta)^* \equiv \eta^* \theta^* = -\theta^* \eta^* \quad (6.27)$$

as this implies $(\theta\eta)^*(\theta\eta) = \theta^* \theta \eta^* \eta$. We furthermore define the integral measure over a complex pair of Grassmann variables as

$$\int d\theta d\theta^* \theta^* \theta = 1. \quad (6.28)$$

In other words, we define this as if we can treat θ and θ^* as independent variables.

Let's now do our first Gaussian Grassman integral (where b is commuting):

$$\int d\theta^* d\theta e^{-\theta^* b \theta} = \int d\theta^* d\theta (1 - \theta^* b \theta) \quad (6.29)$$

$$= - \int d\theta d\theta^* (1 - \theta^* b \theta) \quad (6.30)$$

$$= b \quad (6.31)$$

Now, if θ had been an ordinary complex number, this integral would have instead come out to be

$$\left(\int d\theta^* d\theta e^{-\theta^* b \theta} \right)_{\text{commuting}} = \frac{2\pi}{b} \quad (6.32)$$

The 2π doesn't mean much – the crucial difference here is that the factor of b is in the *denominator* in the normal case, rather than in the numerator as for Grassman variables. This will turn out to be of physical significance.

Now consider an integral of the form

$$\int \prod_i d\theta_i d\theta_i^* f(\theta, \theta^*), \quad (6.33)$$

where $\theta'_i = U_{ij} \theta_j$ for a unitary matrix U . We can use (6.26) to find

$$\int \prod_i d\theta_i d\theta_i^* f(\theta, \theta^*) = (\det U)(\det U)^* \int \prod_i d\theta'_i d\theta'^*_i f(\theta, \theta^*) = \int \prod_i d\theta'_i d\theta'^*_i f(\theta, \theta^*), \quad (6.34)$$

and thus the measure of Grassman integration is invariant under unitary changes of variables, which is perhaps something of a relief. Finally, consider a Hermitian matrix B_{ij} and consider the integral

$$\int \prod_i d\theta_i^* d\theta_i \exp(-\theta_i^* B_{ij} \theta_j) \quad (6.35)$$

As it is Hermitian, we can diagonalize it with a unitary transformation. Call its eigenvalues b_i : we then find

$$\int \prod_i d\theta_i^* d\theta_i \exp(-\theta_i^* B_{ij} \theta_j) = \int \prod_i d\theta_i^* d\theta_i \exp\left(-\sum_i \theta_i^* b_i \theta_i\right) = \int \prod_i d\theta_i^* d\theta_i \exp\left(-\sum_i \theta_i^* b_i \theta_i\right) \quad (6.36)$$

$$= \prod_i b_i = \det B \quad (6.37)$$

Again, in normal integration this determinant would have been in the *denominator*. We almost have everything we need: the final thing that we require is a Gaussian integral in the presence of a linear term, i.e. something of the form

$$I[\eta, \eta^\dagger] = \int \prod_i d\theta_i^* d\theta_i \exp(-\theta^\dagger \cdot B \cdot \theta + \eta^\dagger \theta + \theta^\dagger \eta) \quad (6.38)$$

where η is an external vector of Grassman numbers, and where I am moving to an obvious matrix notation for θ . We can shift the integration variables $\theta \rightarrow \theta + B^{-1}\eta$, $\theta^\dagger \rightarrow \theta^\dagger + \eta^\dagger(B^{-1})$ (recall we set up Grassmann integration to be invariant under shifts) to find that this simply becomes

$$\boxed{I[\eta, \eta^\dagger] = (\det B) \exp(\eta^\dagger B^{-1} \eta)} \quad (6.39)$$

This is perhaps the Fundamental Theorem of Quantum Field Theory With Fermions: it is the basic building block for constructing Feynman rules for fermions from path integration over Grassman numbers. It is *exactly* the same as for commuting numbers, except that the determinant is in the numerator rather than in the denominator.

6.3 Fermions in the path integral

We can now use this central identity to do a path integral over fermions. The basic idea is exactly the same as for scalars. Let us begin by considering the free theory of the Dirac fermion:

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi} (i\not{\partial} - m) \psi \quad (6.40)$$

We now understand a bit better what $\psi(x)$ is: it is a Grassman-valued function of spacetime. Another way to say this is the following: pick some set of functions $f_n(x)$ that provide a complete basis of functions over spacetime (e.g. plane waves are fine). Then we have

$$\psi(x) = \sum_n \psi_n f_n(x) \quad (6.41)$$

where each ψ_n is a Grassman number but the $f_n(x)$ are normal commuting functions.

Now that we understand this, we can formally construct the generating functional for the free Dirac theory:

$$Z_0[\eta, \bar{\eta}] = \int [\mathcal{D}\bar{\psi} \mathcal{D}\psi] \exp\left(iS[\psi, \bar{\psi}] + i \int d^4x (\bar{\eta}(x)\psi(x) + \bar{\psi}(x)\eta(x))\right) \quad (6.42)$$

Here η and $\bar{\eta}$ are anticommuting sources for the Dirac field. Note that there are some extra minus signs to stress about: if we differentiate from the left (which is a convention that we will use), then we have

$$\frac{\delta}{\delta\eta(x)} \int d^4y (\bar{\eta}(y)\psi(y) + \bar{\psi}(y)\eta(y)) = -\bar{\psi}(x) \quad (6.43)$$

$$\frac{\delta}{\delta\bar{\eta}(x)} \int d^4y (\bar{\eta}(y)\psi(y) + \bar{\psi}(y)\eta(y)) = +\psi(x) \quad (6.44)$$

$$(6.45)$$

where the minus sign in the first one comes from anticommuting the two fields to put to η on the left before taking the derivative. By arguments that are precisely the same as in the scalar field case, we may now conclude that the time-ordered correlation function for the fermion field is

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle = Z_0[0]^{-1} \left(-i\frac{\delta}{\delta\bar{\eta}(x_1)} \right) \left(+i\frac{\delta}{\delta\eta(x_2)} \right) Z_0[\eta, \bar{\eta}] \Big|_{\eta, \bar{\eta}=0} \quad (6.46)$$

Let us now evaluate the free theory partition function. From using the identity (6.39) and the quadratic free fermion action, we find

$$\boxed{Z_0[\eta, \bar{\eta}] = (\det(i\cancel{\not{\partial}} - m)) \exp \left(- \int d^4x d^4y \bar{\eta}(x) S_F(x, y) \eta(y) \right)} \quad (6.47)$$

where S_F is formally defined as the inverse of the Dirac operator:

$$(i\cancel{\not{\partial}} - m) S_F(x, y) = i\delta^{(4)}(x - y) \cdot \mathbf{1}_{4 \times 4} \quad (6.48)$$

Of course this expression is better written in Fourier space, where we have

$$(\cancel{\not{p}} - m) S_F(p) = i \quad S_F(p) = \frac{i}{\cancel{\not{p}} - m} \quad (6.49)$$

where the matrix in the denominator just means “matrix inverse” as usual. From this expression we can now construct the two-point function of the fermion field. Plugging this expression into (6.46) and taking derivatives, we see that we bring down a factor of S_F : being very careful about the signs, we find

$$\langle 0|T\psi(x_1)\bar{\psi}(x_2)|0\rangle = S_F(x_1, x_2) = \int \frac{d^4p}{(2\pi)^4} \frac{i \exp^{-ip \cdot (x_1 - x_2)}}{\cancel{\not{p}} - m + i\epsilon} \quad (6.50)$$

where I have put back the $i\epsilon$ for the same reason that it was there previously. This is an expression with which you are familiar; again, we have now computed it from the path integral.

I will now take a second to discuss the business of the determinant being in the numerator in (6.47). Recall way at the beginning that I pointed out that the zero-point energy for the Dirac field had the opposite sign as for a scalar? This zero-point energy can actually be computed from the determinant, and the fact that it has the opposite sign is captured (in the path integral) by the fact that the determinant is in the numerator rather than the denominator. See Chapter II.5 of [3] for more about this.

6.4 Feynman rules for fermions: an example

I would now like to work out the Feynman rules in an example involving fermions. For simplicity let's revisit the scalar Yukawa theory, with a fermion of mass m , a scalar of mass M , and an interaction vertex:

$$S[\psi, \bar{\psi}, \phi] = \int d^4x \left(\bar{\psi}(i\cancel{\not{\partial}} - m)\psi + \frac{1}{2}(\partial\phi)^2 - \frac{M^2}{2}\phi^2 + g\bar{\psi}\phi\psi \right) \quad (6.51)$$

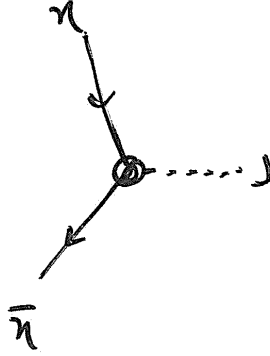


Figure 16: Sample diagram involving fermions

As usual, we do this by expanding the interacting partition function in powers of g in terms of the free theory partition function:

$$Z[\eta, \bar{\eta}, J] = Z_0[0] \exp \left(+ig \int d^4x \left(-i \frac{\delta}{\delta J(x)} \right) \left(+i \frac{\delta}{\delta \eta_a(x)} \right) \left(-i \frac{\delta}{\delta \bar{\eta}_a(x)} \right) \right) Z_0[\eta, \bar{\eta}, J] \quad (6.52)$$

where we sum over the spinor indices a (which I have briefly restored) and where Z_0 is the sum over both the scalar and the fermion part of the free theory partition function:

$$Z_0[\eta, \bar{\eta}, J] = \exp \left(- \int d^4x d^4y \bar{\eta}(x) S_F(x, y) \eta(y) \right) \exp \left(- \frac{1}{2} \int d^4x d^4y J(x) D_F(x, y) J(y) \right) \quad (6.53)$$

It should be fairly clear that this works pretty much as you would expect: you just expand everything and can associate a picture with term in the perturbation expansion. For example, one term arising from the perturbation expansion is

$$D_1 = (ig) \int d^4x d^4y d^4z d^4w (\bar{\eta}(x) S_F(x, y) S_F(y, z) \eta(z)) D_F(y, w) J(w) \quad (6.54)$$

which corresponds to Figure 16, where we draw a solid line to indicate a fermion and a dotted line to indicate a scalar.

I am not going to go into much detail here, as I believe the mechanics have been worked out in your QED course. Everything about LSZ, etc. goes through as before, where you just have to worry about all the spinor indices and be careful with signs, but all else is as you expect. There are only two real wrinkles here.

1. Fermion lines are *oriented*: they have arrows on them. This is not because they are fermions, but because they are complex fields, and so the particles are different from anti-particles: thus we must establish a convention for them. The convention I use is that the arrow points *towards the source* $\bar{\eta}$, which you recall is the source for the field ψ . It is also important that the interaction vertex has one arrow pointing towards the interaction and one pointing away.
2. Closed fermion *loops* come with a minus sign. There are various ways to see this: one way is to realize that the determinant basically captures the physics of closed fermion loops, and its logarithm contributes with the opposite sign as in the commuting case. A more pedestrian way is through direct brutal evaluation of the signs involved. Consider the diagram in Figure 17:

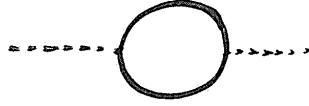


Figure 17: Sample diagram to illustrate minus signs in closed loops of fermions

This has a fermion loop: let's evaluate the sign structure of (6.52), doing the abbreviation

$$\delta_i \equiv \frac{\delta}{\delta\eta(x_i)} \quad \bar{\delta}_i \equiv \frac{\delta}{\delta\bar{\eta}(x_i)} \quad (6.55)$$

we have a term of the form

$$\int dx_1 dx_2 \delta_1 \bar{\delta}_1 \delta_2 \bar{\delta}_2 (\bar{\eta} \cdot S \cdot \eta) (\bar{\eta} \cdot S \cdot \eta) \quad (6.56)$$

Now let us look at the pattern of contractions of derivatives: if we insist that the pattern be such that the propagators form a closed loop, we always need to move an odd number of Grassman derivatives through each other. This gives us a minus sign (and is the same physics behind the determinant).

Let me then summarize with the new physics for the Feynman rules in *momentum space* for fermions. I have not dotted all the i 's⁸ and crossed all the t 's in deriving them, but I leave that to you as an exercise.

- Every fermion propagator comes with a factor of $\frac{i}{\not{p}-m}$. Each propagator has an arrow on it that denotes the flow of charge.
- For the Yukawa theory, every vertex comes with a factor of ig (for more general theories, it is easy to figure out what the right vertex factor should be by staring at the interaction term).
- Every closed fermion loop comes with a minus sign.

I close the section on fermions with a final remark: the difficulty of doing quantum field theory really comes from the loops. However the fact that boson and fermion loops contribute with the opposite sign suggests a tantalizing possibility: could we make them perhaps cancel each other out? If we could argue that they always canceled, then maybe we wouldn't need to calculate them at all, and we could have stopped this course after the first week. This would be a ridiculous coincidence, *unless* you could argue that there was a symmetry that enforces it. This symmetry would seem to require the bosons and fermions to have all the same masses and charges and basically the same couplings. Indeed, this is the idea behind *supersymmetry*, which is a beautiful and important idea that Nature does not seem to be exploiting as much as it could be. B. Hoare will teach a course on this next term.

⁸Ha ha.

7 Abelian Gauge Theories

Now that we have understood fermions, we will move on to gauge theories. We warm up with Abelian gauge theories before moving on to the full glory and beauty of non-Abelian gauge theories for the final part of the course.

7.1 Gauge invariance

We will now first understand what a gauge symmetry is, starting from the simplest example. First consider a theory with a global $U(1)$ symmetry: it is nice to consider the Dirac theory to start,

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi}(x)(i\cancel{\partial} - m)\psi(x) \quad (7.1)$$

This theory is clearly invariant under a global $U(1)$ phase rotation, which is

$$\psi(x) \rightarrow \psi'(x) = e^{i\Lambda}\psi(x) \quad \bar{\psi}(x) \rightarrow e^{-i\Lambda}\bar{\psi}(x) \quad (7.2)$$

I emphasize that the symmetry parameter Λ here is a constant in spacetime, which is why we call it a *global* symmetry. I also want to emphasize that two field configurations that are *related* by the symmetry are both *physical field configurations*. If ψ represents something that we integrate over in the path integral, then so does ψ' .

Now here is an idea, which I simply state without a huge amount of motivation: let's do something different. Let us instead *demand* that we have a theory that is invariant under a symmetry where the symmetry parameter *varies* in spacetime:

$$\boxed{\psi(x) \rightarrow \psi'(x) = e^{i\Lambda(x)}\psi(x)} \quad (7.3)$$

This is called a *local* symmetry, or a *gauge symmetry*. As I will explain later, it should probably really be called a *gauge redundancy*, but I will not use that language in this course and continue to call it a gauge symmetry. Things that are invariant under this symmetry are called *gauge-invariant*.

Now, let's understand what is and is not gauge-invariant. For example: $\psi(x)$ itself is *not* gauge invariant, but it transforms under gauge transformations in a simple way (by an overall rotation).

The mass term

$$m\bar{\psi}'(x)\psi'(x) = m\bar{\psi}(x)e^{-i\Lambda(x)}e^{i\Lambda(x)}\psi(x) \rightarrow m\bar{\psi}(x)\psi(x) \quad (7.4)$$

clearly *is* gauge invariant. Note that it would not have been if the field and its conjugate were evaluated at different points, as then the gauge parameter would not have canceled.

Now how about the derivative term in the action? Now we run into an issue that has to do with the local character of the gauge transformation: note that if we try to take a derivative we find

$$\partial_\mu\psi'(x) = \partial_\mu\left(e^{i\Lambda(x)}\psi(x)\right) = e^{i\Lambda(x)}(i\partial_\mu\Lambda + \partial_\mu)\psi(x) \quad (7.5)$$

Something very bad has happened because $\Lambda(x)$ depends on space, we have picked up an extra term $\partial_\mu\Lambda$. There is no obvious way to get rid of this, and thus we conclude that the Dirac action as written (7.12) is *not* gauge invariant.

How do we fix this? A useful analogy for GR-educated students such as yourselves is to remember that we had a similar problem when we tried to take a derivative of a tensor and found that the resulting object wasn't a tensor any more, in that it had a lousy transformation under coordinate changes. To fix that, we had to invent a new kind of derivative, the covariant derivative.

This is a similar problem: the derivative of ψ no longer has nice gauge transformation properties. To fix it, we will do the same thing that we did previously: we define a new object, called the *gauge covariant derivative*:

$$\boxed{D_\mu \psi \equiv (\partial_\mu + ieA_\mu) \psi} \quad (7.6)$$

where I have introduced a new field A_μ , called the *gauge field* or the *gauge connection*. (GR students may be tempted to compare it to the Christoffel connection: this analogy will get even closer when we do non-Abelian gauge theories soon). I have also chosen to take out a factor e for later convenience.

We will *demand* that the gauge-covariant derivative of ψ has a nice gauge transformation property, i.e. for invariance purposes let us demand:

$$D_\mu \psi \rightarrow D'_\mu \psi' = e^{i\Lambda(x)} D_\mu \psi, \quad (7.7)$$

without any awkward inhomogenous piece. Let's see what we require A'_μ to do to make this happen. Expanding out we have

$$D'_\mu \psi' = (\partial_\mu + ieA'_\mu) e^{i\Lambda(x)} \psi(x) = e^{i\Lambda(x)} (i\partial_\mu \Lambda + ieA'_\mu + \partial_\mu) \psi(x) \quad (7.8)$$

$$= e^{i\Lambda(x)} (\partial_\mu + ieA_\mu) \psi(x) \quad (7.9)$$

where the last equality is our demand. We see that this will work out if and only if:

$$\boxed{A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \Lambda(x) \quad \psi'(x) = e^{i\Lambda(x)} \psi(x)}, \quad (7.10)$$

where I have also written down the transformation property of ψ , as the two are related. This is the *gauge-transformation of A_μ* : if it transforms in this way, then our gauge-covariant derivative does indeed do what we want it to do. Note that now the following term:

$$\bar{\psi}(x) \gamma^\mu D_\mu \psi(x) \rightarrow \bar{\psi}'(x) \gamma^\mu D'_\mu \psi'(x) = \bar{\psi}(x) e^{-i\Lambda(x)} \gamma^\mu e^{+i\Lambda(x)} D_\mu \psi(x) = \bar{\psi}(x) \gamma^\mu D_\mu \psi(x) \quad (7.11)$$

is gauge-invariant, and thus we have succeeded in taking a derivative in a gauge-invariant way. With this, we see that we can write a fully gauge-invariantized-Dirac-action:

$$S[\psi, \bar{\psi}, A]_{\text{Dirac}} = \int d^4x \bar{\psi}(x) (i\mathcal{D} - m) \psi(x) \quad (7.12)$$

where we just promote the partial to a gauge-covariant derivative. To do this, we had to invent another field called A_μ . This is the price of gauge-invariance.

But now that we have this field A_μ , we may ask what more we can do with it. To see this, let's first note that $D_\mu \psi$ has a nice gauge-transformation property, and thus so will $D_\mu D_\nu \psi$. It thus makes sense to consider the *commutator* of the two derivatives acting on ψ

$$[D_\mu, D_\nu] \psi = [\partial_\mu, \partial_\nu] \psi + ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]) \psi \quad (7.13)$$

$$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \psi \quad (7.14)$$

Notice that this commutator of derivatives is (when acting on ψ) actually not a derivative at all: it is just a number multiplying ψ . Let's call this number $ieF_{\mu\nu}$, i.e.

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \quad (7.15)$$

$F_{\mu\nu}$ is called the *field-strength* of the gauge field. Now, purely abstractly, we know that $F_{\mu\nu}$ must be gauge-invariant. That's because we have

$$[D_\mu, D_\nu] = ieF_{\mu\nu} \quad (7.16)$$

Now consider acting this equation on ψ and then doing a gauge-transformation on both sides: we know how both sides transform

$$e^{i\Lambda(x)}[D_\mu, D_\nu]\psi = ieF_{\mu\nu}e^{i\Lambda(x)}\psi \quad (7.17)$$

and we can now cancel the $e^{i\Lambda(x)}$, which immediately implies that $F_{\mu\nu}$ is gauge invariant. Of course we could also simply directly use the gauge transformation property of A_μ (7.10) directly, but this more abstract viewpoint will be useful when we get to non-Abelian gauge theories.

It is not hard to convince yourself that random derivatives of A_μ , i.e. $\partial_\mu A_\nu$ by itself, are actually not gauge-invariant. So let us now try to write down the most general renormalizable gauge-invariant action involving our new friend A_μ and $\psi(x)$. It turns out the most general such action is

$$S[\psi, \bar{\psi}] = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(x)(i\not{D} - m)\psi(x) + \theta\epsilon^{\alpha\beta\mu\nu}F_{\alpha\beta}F_{\mu\nu} \right) \quad (7.18)$$

The first term is gauge-invariant since $F_{\mu\nu}$ is gauge invariant. We have discussed the second term extensively. The third term looks weird, perhaps – it is weird for two distinct reasons: 1. It has an ϵ symbol, which means that it breaks parity and time reversal, and 2. It is a total derivative, and so does not seem to obviously contribute to the equations of motion – but see *The Uses of Instantons* in S. Coleman’s *Aspects of Symmetry*. We can, however, set θ to 0 if we demand that our theory be parity-invariant⁹ and from now on we will not worry about it.

How about a mass term for the gauge field, e.g.

$$M^2 A_\mu A^\mu \quad (7.19)$$

It is easy to see that this is not gauge-invariant, and thus one says that gauge-invariance forbids bare mass terms for gauge fields. Thus we conclude that the most general gauge-invariant and P -invariant action with a gauge field and a single fermion is just

$$\boxed{S[\psi, \bar{\psi}, A]_{\text{QED}} = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} + \bar{\psi}(x)(i\not{D} - m)\psi(x) \right)} \quad (7.20)$$

This is the action of QED. Note that we have been led to it purely from symmetry principles: *there is nothing else that we could have written down.*

7.2 Some classical aspects of Abelian gauge theory

For a little while, let us focus on the theory given by just the first part of the action above. This turns out to be ordinary Maxwell electrodynamics:

$$S[A]_{\text{Maxwell}} = \int d^4x \left(-\frac{1}{4}F^{\mu\nu}F_{\mu\nu} \right) \quad (7.21)$$

Let’s figure out the classical equations of motion: varying this action with respect to A_μ we find that these equations of motion are just

$$\partial_\mu F^{\mu\nu} = 0 \quad (7.22)$$

If we pick a time direction and write this down in Lorentz-non-invariant notation $F^{0i} = E^i$, $B^i = \frac{1}{2}\epsilon^{ijk}F_{jk}$ then these are simply Maxwell’s equations, and thus this is the Lagrangian for free EM with no matter.

⁹Ok, this is not strictly true: more formally, P flips $\theta \rightarrow -\theta$; however it turns out that when θ is correctly normalized it is a periodic variable with periodicity 2π , which means that there is *two* P -invariant points, one with $\theta = 0$ and one with $\theta = \pi$, as $-\pi + 2\pi = \pi$. This seems like a ridiculous technicality that nobody should ever worry about, except that actually a lot of beautiful physics associated with something called a “topological insulator” arises from the $\theta = \pi$ point.

Now, we should note an interesting fact: recall the Klein-Gordon equation of motion for a free scalar field

$$\partial^2\phi = 0 \tag{7.23}$$

This defines what is called a *well-defined Cauchy problem*, which means that if you pick a time $t = 0$ and specify initial data $\phi(t = 0, \vec{x})$ as well as $\partial_t\phi(t = 0, \vec{x})$, then you can use the equations of motion to propagate ϕ forwards in time unambiguously. This is nice.

The fundamental degree of freedom for the Maxwell equation is $A_\mu(x)$, so this is the analog of $\phi(x)$. The above nice time evolution properties of the Klein-Gordon equation are *not* satisfied by the Maxwell equation. This can be understood from staring at the components of the equation, but there is an indirect route: note that if A_μ is a solution to the equations of motion, then so is $A_\mu - \frac{1}{e}\partial_\mu\Lambda(x)$, where Λ is any arbitrary function of spacetime. If we pick $\Lambda(t, \vec{x})$ to be a function that vanishes near $t = 0$ but is different at late t , it is clear that the same initial data at $t = 0$ can lead to two completely different A_μ 's at late time, where they differ by a gauge transformation.

What do we do with this? It looks like our theory has no predictive power, in that knowing what happens at $t = 0$ does not fix what happens at late t . We can save our classical theory by making the following assertion:

Things that are not gauge-invariant are *not physical*.

So it is okay that we cannot unambiguously solve for their time evolution. It is only things that are gauge-invariant – in this case, the components of $F_{\mu\nu}$ – that are physical, and you know from your elementary EM course that there is absolutely no issue solving for their time evolution.

Let's be a bit more explicit. Writing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we find

$$\partial_\mu(\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \tag{7.24}$$

As mentioned earlier, this is not a well-defined problem for time evolution of A_μ . We need to *fix a gauge*: one choice is Lorenz gauge:

$$\partial_\mu A^\mu = 0 \tag{7.25}$$

(You can show that any A_μ can be placed in this form by a gauge transformation). If we do this, then we find the equation

$$\partial_\mu\partial^\mu A_\nu = 0, \tag{7.26}$$

which is now a perfectly nice wave equation that propagates all components of A_μ in time. It is also a wave equation with no mass term: in Fourier space, we get an equation that looks like:

$$(\omega^2 - k^2)A_\nu = 0 \tag{7.27}$$

and thus the on-shell energy of a single photon is $\omega_k = k$: the photon is massless.

However, it is still not true that all components are physical, as there is a *residual gauge invariance* in Lorenz gauge. I will not review here how this works, but you recall from elementary EM that for a plane wave with momentum k_μ it is the *transverse* components of A_μ to the momentum that are physical, and thus there are two physical polarizations. (One way to check this is to note that only those contribute to the field strength tensor; more formally you can find a way to fix the residual gauge invariance in Lorenz gauge).

With this classical understanding under control, we now turn to the quantum theory, where all these subtleties turn out to play a crucial role.

7.3 Quantizing QED

We turn now to the quantum theory. The new thing to do is to *quantize* the Maxwell action, i.e. we want to perform the following path integral:

$$Z = \int [\mathcal{D}A] \exp \left(i \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right) \quad (7.28)$$

We will ultimately be interested in quantizing the theory with the fermions also included, but first we will focus on this. As we understand in detail by now, the propagator is always equal to the inverse of the operator appearing in the quadratic part of the Lagrangian: let us thus write this in a useful way:

$$S[A] = \int d^4x \left(-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) \right) \quad (7.29)$$

$$= \int d^4x \frac{1}{2} (A_\nu \partial_\mu \partial^\mu A_\nu - A_\nu \partial^\nu \partial_\mu A^\mu) \quad (7.30)$$

$$= \int d^4x \frac{1}{2} (A_\mu (\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu) \quad (7.31)$$

Now whenever we do a functional integral, we want to invert the differential operator that appears in the quadratic part of the action. Thus we seek to find a photon Feynman propagator $D_{\mu\nu}^F$ that satisfies:

$$(\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) D_{\nu\rho}^F(x, y) = i \delta_\rho^\mu \delta^{(4)}(x - y) \quad (7.32)$$

There is just one problem with this: no such $D_{\nu\rho}^F$ exists, because the differential operator *has no inverse*. This is perhaps clear if we go to momentum space, when this equation becomes

$$(p^2 \eta^{\mu\nu} - p^\mu p^\nu) D_{\mu\nu}^F(p) = -i \delta_\nu^\mu \quad (7.33)$$

But the matrix on the left-hand side has many zero eigenvalues: in fact, consider acting with it on any test function of the form $p_\mu \alpha(p)$:

$$(p^2 \eta^{\mu\nu} - p^\mu p^\nu) p_\mu \alpha(p) = (p^2 p^\nu - p^2 p^\nu) \alpha(p) = 0 \quad (7.34)$$

Thus the operator has a non-trivial kernel and cannot be inverted.

Physically, it should be clear that the issue is arising from the gauge transformations. The point is that any field configuration that is *pure gauge*

$$A_\mu = \frac{1}{e} \partial_\mu \Lambda \quad (7.35)$$

has field strength $F_{\mu\nu} = 0$, and thus has vanishing action. Thus when performing the path integral over functions in this “pure gauge” direction, the action does not oscillate and nothing suppresses their contribution to the path integral, which thus ends up diverging badly. This divergence is manifesting itself in the non-invertibility of the differential operator.

Clearly we are doing something wrong. Recall that things that are not gauge-invariant are not physical: thus somehow we are integrating over a bunch of non-physical stuff. Somehow we need to rearrange the integral so that we integrate over only the physical stuff.

We do this with a trick called the *Faddeev-Popov procedure*, which is basically the correct quantum analog of the gauge-fixing we did in the classical theory. Let $G(A)$ be a function that we want to set to zero as a gauge-fixing: e.g. if we were doing Lorenz gauge we would have

$$G(A) = \partial_\mu A^\mu \quad \text{Lorenz gauge} \quad (7.36)$$

Let us now introduce the concept of a *functional delta function*: this is just what it sounds like, it is a map from functions to numbers that satisfies the properties:

$$\delta[\phi] = 0, \quad \phi(x) \neq 0 \quad (7.37)$$

$$\int [\mathcal{D}\phi] \delta[\phi] = 1 \quad (7.38)$$

You can think of a functional delta function as being an infinite product of delta functions, one at each point in space. We want to somehow stick a functional delta function, a factor of $\delta(G(A))$, inside the functional integral. We do this by first defining the gauge transformed field A^Λ :

$$A_\mu^\Lambda(x) \equiv A_\mu(x) - \frac{1}{e} \partial_\mu \Lambda \quad (7.39)$$

and then considering the following very fancy way to write the number 1:

$$1 = \int [\mathcal{D}\Lambda] \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) \quad (7.40)$$

Note the factor of the functional determinant: this is just the Jacobian factor that you find when changing variables in normal integrals, i.e. recall in one dimension we have the identity:

$$1 = \int dy \delta(y) = \int dx \delta(y(x)) \frac{dy}{dx} \quad (7.41)$$

This is the same idea.

In any case, we now insert 1 in this fancy form into the functional integral, to obtain:

$$Z = \int [\mathcal{D}A] [\mathcal{D}\Lambda] \delta(G(A^\Lambda)) \det\left(\frac{\delta G(A^\Lambda)}{\delta \Lambda}\right) e^{iS[A]} \quad (7.42)$$

We now need to pick a convenient gauge-fixing function: we will pick the following:

$$G_\omega(A) = \partial_\mu A^\mu - \omega(x) \quad (7.43)$$

where ω is some function. It does not matter what function it is: this picking of a gauge should not affect anything at all. However something nice about this choice is that we see that

$$\frac{\delta G(A^\Lambda)}{\delta \Lambda} = \frac{\delta}{\delta \Lambda} \left(\partial_\mu \left(A^\mu + \frac{1}{e} \partial^\mu \Lambda \right) - \omega \right) = \frac{1}{e} \partial^2 \quad (7.44)$$

And thus this determinant is independent of everything! Thus we can take it outside the functional integral (and basically forget about it, at least in the Abelian case). We now find

$$Z_\omega = \det\left(\frac{1}{e} \partial^2\right) \int [\mathcal{D}\Lambda] \int [\mathcal{D}A] \delta(G_\omega(A^\Lambda)) e^{iS[A]} \quad (7.45)$$

where I have put some ω 's in places to remind us that various things depend on ω . Now note that the action is invariant, so we can write $S[A] = S[A^\Lambda]$. Similarly, the integration measure is invariant (as it is just a shift), and we can thus write $[\mathcal{D}A] = [\mathcal{D}A^\Lambda]$. But now A by itself doesn't appear anywhere in the integral! So let us just rename $A^\Lambda \rightarrow A$, which we are clearly free to do, as its just a name. We then find

$$Z_\omega = \det\left(\frac{1}{e} \partial^2\right) \int [\mathcal{D}\Lambda] \int [\mathcal{D}A] \delta(G_\omega(A)) e^{iS[A]} \quad (7.46)$$

Now all of the integrals over Λ have completely factored out – they just contribute an overall prefactor that we will just call \mathcal{N} . So we thus have

$$Z_\omega = \mathcal{N} \int [\mathcal{D}A] \delta(\partial_\mu A^\mu - \omega(x)) e^{iS[A]} \quad (7.47)$$

Now we will do one last thing – note that my notation makes it seem that Z_ω depends on ω , but actually it does not, as this is just a gauge-fixing condition. It is thus completely fine to average over all ω with a Gaussian weight with width ξ . (Why not?). We then arrive at the following form of the partition function:

$$Z = \mathcal{N}' \int [\mathcal{D}\omega][\mathcal{D}A] \exp\left(-i \int d^4x \frac{1}{2\xi} \omega^2\right) \delta(\partial_\mu A^\mu - \omega(x)) e^{iS[A]} \quad (7.48)$$

The point of doing this possibly rather Baroque step is to make it possible to solve the delta function constraint in a simple way. Now that we integrate over all ω , we can interchange the order of integration and see that actually the delta function sets $\omega = \partial_\mu A^\mu$. Thus we find the following *gauge-fixed* form for the path integral:

$$\boxed{Z = \mathcal{N}' \int [\mathcal{D}A] \exp\left(iS[A] - \int d^4x \frac{1}{2\xi} (\partial_\mu A^\mu)^2\right)} \quad (7.49)$$

The point of all of this was to modify the effective form of the Maxwell action to the following:

$$S_{\text{FP}}[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{1}{2\xi} (\partial_\mu A^\mu)^2\right) \quad (7.50)$$

In other words, we have simply *added a gauge-fixing term to the action*. Note that the slick way in which we did it guarantees that everything is okay in the quantum theory. If we integrate by parts as before this action becomes:

$$S_{\text{FP}}[A] = \int d^4x \frac{1}{2} \left(A_\mu \left(\partial^2 \eta^{\mu\nu} - \left(1 - \frac{1}{\xi}\right) \partial^\mu \partial^\nu \right) A_\nu \right) \quad (7.51)$$

Thus the kinetic term has been modified, and of course now we can now invert the propagator. We go to Fourier space, and the equation to solve is then:

$$\left(-p^2 \eta_{\mu\nu} + \left(1 - \frac{1}{\xi}\right) p_\mu p_\nu\right) D_F^{\nu\rho}(p) = i\delta_\mu^\rho \quad (7.52)$$

This sort of computation comes up again and again in field theory, so in this case I will work it out: consider the most general form of D_F . It is a tensor, and the only two tensor structures are $\eta^{\mu\nu}$ and $p^\mu p^\nu$, so we the answer must be of the form

$$D_F^{\nu\rho}(p) = A(p) \eta^{\nu\rho} + B(p) \frac{p^\nu p^\rho}{p^2} \quad (7.53)$$

where I have taken out some factors of p to make the final thing look nicer. Now we plug this ansatz into the equation and expand. We find

$$-p^2 A(p) \delta_\mu^\rho + p_\mu p^\rho \left(A(p) \left(1 - \frac{1}{\xi}\right) - \frac{B(p)}{\xi} \right) = i\delta_\mu^\rho \quad (7.54)$$

(Note taking away the gauge-fixing term corresponds to $\xi \rightarrow \infty$, in which case there is clearly no solution). We find that $A = -\frac{i}{p^2}$, $B = -\frac{i}{p^2}(\xi - 1)$, and thus:

$$\boxed{D_F^{\nu\rho}(p) = -\frac{i}{p^2} \left(\eta^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right)} \quad (7.55)$$

This is the Feynman propagator for photons, carefully derived.

$$\mu \text{ --- } \nu = \frac{-i}{p^2} \left(\eta^{\mu\nu} - (1-\xi) \frac{p^\mu p^\nu}{p^2} \right)$$

$$\begin{array}{c} \swarrow \\ \text{---} \\ \searrow \end{array} \mu = -ie \gamma^\mu$$

Figure 18: New Feynman rules involving photons

What is ξ ? The truth is: it is a gauge-fixing parameter, and so it *does not matter*. You can set it to any value you like, and you have to get the same answer for any calculation. This turns out to be guaranteed by the Ward identity in QED – basically dotting k^μ into any amplitude has to annihilate it. Thus people with truly formidable calculational powers will keep it arbitrary, and its cancellation at the end is a good check on the computation.

People (like me) with less formidable calculational powers will often find it convenient to set it to 1 (this is called Feynman gauge) as this makes the photon propagator have fewer terms in it.

This is it! Now you have carefully derived the photon propagator. This is the only tricky thing: from here we can rederive the full machinery of the Feynman rules. If we don't have the fermion then the theory is free; if we do have the fermion then the action is

$$S[\psi, \bar{\psi}, A]_{\text{QED}} = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(x)(i\gamma^\mu(\partial_\mu + ieA_\mu) - m)\psi(x) \right) \quad (7.56)$$

and thus we have the fermion Feynman rules that we wrote down last time and the following new rules for the photon:

- The photon propagator is $-\frac{i}{p^2} \left(\eta^{\mu\nu} - (1-\xi) \frac{p^\mu p^\nu}{p^2} \right)$
- The photon-electron vertex is $-ie\gamma^\mu$.

(If you like, you can carefully derive the interaction vertex by doing the usual business with the sources, etc – no new effects enter from the gauge field).

It is important to note that the interaction vertex *came from the covariant derivative*, and thus we were not free to fix its form. This is an underlying principle: gauge symmetry fixes the possible interactions that can happen.

In the QED course you have extensively calculated things with these Feynman rules, and now you know where they come from.

7.4 Things that I did not do: canonical quantization and LSZ

In the interests of time, we are now going to press on to non-Abelian gauge theory. Having said that, there are a few things that I simply omitted. The most glaring omission is that I did not carefully discuss canonical quantization for the photon fields. Let me say a few brief words, and you can find more details in Srednicki chapters 54-57, and in Chapter 6 of [D. Tong's lecture notes](#).

You might be tempted to canonically quantize the photon field as follows:

$$A_\mu(x) = \sum_\lambda \int \frac{d^3k}{(2\pi)^3 2\omega_k} \left(\epsilon_\mu^\lambda(\vec{k}) a_{\lambda, \vec{k}} e^{-ik \cdot x} + \epsilon_\mu^{*\lambda}(\vec{k}) a_{\lambda, \vec{k}}^\dagger e^{+ik \cdot x} \right) \quad (7.57)$$

where the λ runs over photon polarizations, the ϵ_μ^λ are polarization vectors, and the a and a^\dagger are creation and annihilation operators, one for each photon polarization. The only issue is that not all of the states here are physical, and we have to deal with this – all ways of doing it cut out two polarizations, leaving us with 2 physical photon polarization. I will not go into this here and refer you to Srednicki, who explains the problem in Coulomb gauge:

$$\partial_i A^i = 0 \quad (7.58)$$

which is different from Lorenz gauge in that it is not Lorentz-invariant. The end effect is that the sum over λ only includes two spatial polarizations that are always perpendicular to \vec{k} . I will say that the Hamiltonian formulation of this theory can get a bit tricky, and that we really sidestepped a lot of these complications by going through the path integral.

Once you accept this, it is not hard to go through LSZ for photons – everything goes through as before as in (3.23), except that there are some factors of ϵ_μ floating around:

$$\prod_{i=1}^n \frac{i}{k_i^2} \prod_{i=1}^{n'} \frac{i}{k_{i'}^2} \langle f | i \rangle = \epsilon_1^\mu \epsilon_2^\mu \cdots \epsilon_{n'}^{*\mu} \langle 0 | A_\mu(k_1) A_\mu(k_2) \cdots A_\mu(k_{n'}) | 0 \rangle \quad (7.59)$$

which tells us the rule for external photon lines: which is that you simply amputate the propagator, and in-states come with a factor of $\epsilon_\mu(k)$ while out-states come with a factor of $\epsilon_\mu^*(k)$.

8 Non-Abelian gauge theories

Now that we understand Abelian gauge theories, we are going to move on to the next and final part of this course: *non-Abelian* gauge theories. They are a powerful and nontrivial generalization of the same basic idea, and play a very important role in the Standard Model.

8.1 Non-Abelian gauge invariance

An abstract way of formulating the discussion above was the following: a gauge transformation in the Abelian case was a map from spacetime to the group $U(1)$, i.e. something like $e^{i\Lambda(x)}$. To be completely concrete, for any field ψ we should specify its *charge* q , and then the gauge transformation acts on the field like

$$\psi(x) \rightarrow e^{iq\Lambda(x)}\psi(x) \quad (8.1)$$

(We actually only discussed things with charge $q = 1$, but it should be clear that you could imagine other values for the charge of a field).

Now we will generalize the whole thing to non-Abelian gauge theory. Consider a (compact, simple) non-Abelian group G : the construction will work for any G , but for concreteness let's take G to be $SU(N)$. (In the back of my mind, I will always be imagining $G = SU(2)$, but if your favorite group is $SU(19)$ or something like that don't let me stop you). We will now try to formulate a theory that is invariant under non-Abelian gauge transformations, which are maps $U(x)$ from spacetime to the group G .

Let's review a little bit of group theory: I denote an abstract set of generators of the group by T^a , where the index a runs over the Lie algebra of the group. Let me denote the dimension of the algebra by $d(G)$ (e.g. $d(SU(N)) = N^2 - 1$). The T^a satisfy the following equation:

$$[T^a, T^b] = if^{abc}T^c \quad (8.2)$$

where the f^{abc} are the *structure constants*. Note that there is the Bianchi identity that all operations obey:

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \quad (8.3)$$

This identity together with the definition of the structure constants implies that they obey:

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0 \quad (8.4)$$

This is called the *Jacobi identity*.

Now there are many different representations for the group, i.e. the abstract T^a can be represented by concrete matrices. Let me denote the actual matrices in the fundamental representation of the group by little t^a . It is of course also true that

$$[t^a, t^b] = if^{abc}t^c \quad (8.5)$$

e.g. for the group $SU(2)$ we have three generators and the $t^i = \frac{1}{2}\sigma^i$ where the σ^i are the Pauli matrices.

Now let us imagine having a fermion ψ that transforms in the fundamental of our $SU(N)$ gauge group: i.e. if $U(x) \in SU(N)$, we want to have the symmetry property:

$$\boxed{\psi(x) \rightarrow \psi'(x) = U(x)\psi(x)}, \quad (8.6)$$

where ψ is now an N -component complex vector, as well as a spinor (so technically it has $4N$ components; we are suppressing all of that information here).

Again, we can imagine writing down the Dirac Lagrangian:

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi} (i\not{\partial} - m) \psi \quad (8.7)$$

It is clear that this is invariant under $\psi(x) \rightarrow U\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x)U^\dagger$ if U is constant; however, if $U(x)$ depends on space, then clearly the derivatives will not work out. The solution is clear: we now need to construct a non-Abelian gauge-covariant derivative.

We will do this as follows: the property that we want for the non-Abelian covariant derivative is

$$D_\mu \psi(x) \rightarrow D'_\mu \psi'(x) = U(x) D_\mu \psi(x) \quad (8.8)$$

in other words, we want the whole thing to transform homogeneously. From the $U(1)$ case, it should be clear that we will need a new object to do this: our new object is going to be the *non-Abelian gauge field*:

$$A_\mu^a(x) \quad (8.9)$$

where a runs over the Lie algebra of the group. The covariant derivative acting on ψ is taken to be

$$\boxed{D_\mu \psi(x) \equiv (\partial_\mu - igA_\mu^a t^a) \psi(x)} \quad (8.10)$$

where g is a number called the *gauge-coupling*, and is analogous to the e that we defined in the $U(1)$ case. The t^a are the group representation matrices in the fundamental: they are in the fundamental because I have taken ψ to be in the fundamental. Now let us see what property we require A_μ^a to have for (8.8) to work out:

$$D'_\mu \psi'(x) = (\partial_\mu - igA_\mu^a t^a) U(x) \psi(x) \quad (8.11)$$

$$= (\partial_\mu U) \psi + U(x) \partial_\mu \psi - igA_\mu^a t^a U(x) \psi(x) \quad (8.12)$$

$$\equiv U(x) (\partial_\mu - igA_\mu^a t^a) \psi(x), \quad (8.13)$$

where the last equality is what we *want* to be true. There is an interesting wrinkle here, because on the last line $U(x)$ is all the way on the left, whereas on the previous line it is always on the *right*, and because of the non-Abelian-ness we cannot commute it through. Nevertheless, we can still solve for A_μ^a to find

$$\partial_\mu U - igA_\mu^a t^a U = U(-igA_\mu^a t^a) \quad \rightarrow \quad \boxed{A_\mu^a(x) t^a = U(x) \left(A_\mu^a(x) t^a + \frac{i}{g} \partial_\mu \right) U^\dagger(x)} \quad (8.14)$$

This is the transformation of the non-Abelian gauge field under a finite gauge transformation parametrized by $U(x)$. Before discussing what it means, it is instructive to also work out the *infinitesimal* version of the gauge transformation. To do this we parametrize $U(x)$ as

$$U(x) = \exp(i\alpha^a(x)t^a) \quad (8.15)$$

and then work to first order in α^a . The fermion transformation law (8.6) is simply

$$\psi'(x) = (1 + i\alpha^a(x)t^a) \psi(x) \quad (8.16)$$

whereas the gauge field one is slightly more intricate:

$$A_\mu^a(x) t^a = \exp(i\alpha^b(x)t^b) \left(A_\mu^a(x) t^a + \frac{i}{g} \partial_\mu \right) \exp(-i\alpha^c(x)t^c) \quad (8.17)$$

$$= A_\mu^a(x) t^a + i[\alpha^b(x)t^b, A_\mu^a t^a] + \frac{1}{g} \partial_\mu \alpha^c t^c \quad (8.18)$$

$$= A_\mu^a(x) t^a - \alpha^b f^{bac} t^c A_\mu^a + \frac{1}{g} \partial_\mu \alpha^c t^c \quad (8.19)$$

Relabeling some indices and equating terms we find that the individual components of the gauge field transform as transform as

$$A'_\mu{}^a(x) = A_\mu^a + f^{abc} A_\mu^b \alpha^c + \frac{1}{g} \partial_\mu \alpha^a \quad (8.20)$$

So, what is going on? Here are a few important points:

1. The last term looks like the transformation of an Abelian gauge field.
2. The first term is different: what it means is that the non-Abelian gauge field points in a direction in group space (i.e. it has an a index), and the non-Abelian transformation rotates the direction that it points in. Let us be a bit more precise about this: A_μ^a is *trying* to be in the adjoint. Recall that the adjoint is a special representation. In a sense it tells us how elements of the Lie algebra themselves transform – i.e. they are a special representation where the dimension is the same as that of the Lie algebra itself, so if Φ is in the adjoint it has $N^2 - 1$ components and I can write Φ^a . The way in which something in the adjoint transforms is:

$$\Phi^a t^a \rightarrow \Phi'^a t^a = U \Phi^a t^a U^\dagger \quad (8.21)$$

This can be understood as a linear transformation on the components Φ^a .

Note from (8.14) now that the gauge field *almost* transforms like this, but not quite because of the inhomogenous bit.

3. Finally, this form should make clear that the transformation of the gauge field has nothing to do with the representation of the field it is acting on. (e.g. our ψ was in the fundamental, but that was just for my notational convenience – it is not important). In general you will have many different fields in different representations of a given group, but you only have one non-Abelian gauge field for all of them.

To emphasize this, let's compute the covariant derivative of a field Φ^a that is in the *adjoint*. In particular, if we denote the representation matrices in the adjoint as T_G^a , then from (8.21) we can deduce that the components of these matrices are themselves the structure constants

$$(T_G^a)_{bc} = i f^{bac} \quad (8.22)$$

Let me explain the notation: a tells me which element of the algebra I am talking about, whereas b and c denote the row and column of the matrix itself. Using this expression, the covariant derivative of a field in the adjoint is then

$$D_\mu \Phi^a = \partial_\mu \Phi^a - ig A_\mu^b (T_G^b)_{ac} \Phi^c \quad (8.23)$$

$$= \partial_\mu \Phi^a + g f^{abc} A_\mu^b \Phi^c \quad (8.24)$$

where in the last equality we used the definition of the representation matrix in the adjoint. In the same way you can construct the covariant derivative of a field in any representation of the gauge group G . (It is interesting to note that actually the action depends only on the Lie algebra, not on the global properties of the group).

4. As a brief aside: I will sometimes write A_μ with no superscript to mean $A_\mu \equiv A_\mu^a t^a$, i.e. it is a matrix-valued (more properly, a Lie-algebra valued) vector field (more properly, a one-form).

8.2 The Yang-Mills field-strength and action

So we have understood how to take non-Abelian covariant derivatives, and we have seen that this requires us to introduce a non-Abelian gauge field A_μ^a . We would now like to build an action out of these gauge

fields. We will do this in pretty much the same way as in the Abelian case: in particular, let's consider the commutator of two covariant derivatives acting on ψ . From the above discussion, we know that under a gauge transformation:

$$[D_\mu, D_\nu]\psi(x) \rightarrow U(x)[D_\mu, D_\nu]\psi(x) \equiv U(x) (-igF_{\mu\nu}^a t^a) \psi(x) \quad (8.25)$$

i.e. there are no pesky derivative terms. Thus this object has nice transformation properties and is a natural thing to think about. We will call it the field-strength tensor, and as the notation suggests, it is only a multiplicative operator acting on $\psi(x)$, not a differential one. Let's work it out:

$$[\partial_\mu - igA_\mu^a t^a, \partial_\nu - igA_\nu^b t^b]\psi = (-ig(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) t^a - g^2 A_\mu^a A_\nu^b [t^a, t^b]) \psi \quad (8.26)$$

$$= (-ig(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) t^a - ig^2 A_\mu^b A_\nu^c f^{bca} t^a) \psi \quad (8.27)$$

Comparing this to the definition, we see that in components the field-strength is

$$\boxed{F_{\mu\nu}^a = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + gA_\mu^b A_\nu^c f^{abc}} \quad (8.28)$$

Importantly, from our previous study we know that under a gauge transformation we have

$$F_{\mu\nu} = F_{\mu\nu}^{\prime a} t^a = U(x) F_{\mu\nu}^a t^a U^\dagger(x) . \quad (8.29)$$

Thus $F_{\mu\nu}$ transforms in the adjoint. As explained above, the infinitesimal version of this transformation is

$$F_{\mu\nu}^{\prime a} = F_{\mu\nu}^a - f^{abc} \alpha^b F_{\mu\nu}^c + \mathcal{O}(\alpha^2) \quad (8.30)$$

It is possible to check these properties explicitly by directly transforming the quantities in (8.28); there is a non-trivial cancellation between the terms that makes this possible, and thus the precise factors of g etc. are important. (The GR-savvy student will recall that the construction here is pretty much how we construct the Riemann tensor in GR).

However, now that we have this object, we can construct a non-Abelian-gauge-invariant action for the gauge field. This is essentially completely unique – just consider taking the trace of the matrix-valued field strength squared.

$$\boxed{S_{\text{YM}}[A] = \int d^4x \left(-\frac{1}{2} \text{tr} (F_{\mu\nu} F^{\mu\nu}) \right)} \quad (8.31)$$

This is the celebrated Yang-Mills action. Let's check its properties: due to (8.29), under a gauge transformation we have

$$\text{tr}(F_{\mu\nu}^{\prime} F^{\prime\mu\nu}) = \text{tr}(U(x) F_{\mu\nu} U^\dagger(x) U(x) F^{\mu\nu} U^\dagger(x)) = \text{tr}(F_{\mu\nu} F^{\mu\nu}) \quad (8.32)$$

and so it is clearly invariant. It's also convenient to work it out in components: we have

$$-\frac{1}{2} \text{tr} (F_{\mu\nu}^a t^a F^{b\mu\nu} t^b) = -\frac{1}{2} F_{\mu\nu}^a F^{b\mu\nu} \text{tr} (t^a t^b) \quad (8.33)$$

From your group theory course, it is always possible to pick the generators of $SU(N)$ so that we have

$$\text{tr} (t^a t^b) = \frac{1}{2} \delta^{ab} \quad (8.34)$$

which is a convention we will use – we then have that the action is

$$S_{\text{YM}}[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right) \quad (8.35)$$

where we sum over all the indices of the gauge field. This is it. Note that since $F_{\mu\nu}^a$ is non-linear in the field A_μ , this is a non-trivial interacting theory already. In the next section we will study the quantum theory – but first, let's study a few classical aspects.

Consider varying this action: we have

$$\delta S_{\text{YM}}[A] = \int d^4x \left(-\frac{1}{2} F^{a\mu\nu} \delta F_{\mu\nu}^a \right) \quad (8.36)$$

$$= \int d^4x \left(-\frac{1}{2} F^{a\mu\nu} (\partial_\mu \delta A_\nu^a - \partial_\nu \delta A_\mu^a + g(\delta A_\mu^b A_\nu^c + A_\mu^b \delta A_\nu^c) f^{abc}) \right) \quad (8.37)$$

$$= \int d^4x (\partial_\mu F^{a\mu\nu} + g f^{abc} A_\mu^b F^{c\mu\nu}) \delta A_\nu^a \quad (8.38)$$

From here we can read off that the equations of motion are

$$\partial_\mu F^{a\mu\nu} + g f^{abc} A_\mu^b F^{c\mu\nu} = 0 \quad (8.39)$$

Note that this equation is actually quite elegant – we are trying to take the derivative of an object $F_{\mu\nu}^a$ that transforms in the adjoint, so this is actually the gauge-covariant divergence $F_{\mu\nu}^a$:

$$\boxed{D_\mu F^{a\mu\nu} = 0} \quad (8.40)$$

This is the classical Yang-Mills equation. Despite the deceptive notation, it's a very nonlinear and messy equation.

We can of course also couple matter to these gauge fields. Let's then recap the rules for making a gauge theory: pick a gauge group G . Pick some matter content (e.g. some fermions ψ). Pick a representation for the fermions to transform in. Then write down the action:

$$S[A, \psi, \bar{\psi}] = \int d^4x \left(-\frac{1}{2} \text{tr}(F^2) + \bar{\psi} (i\not{D} - m) \psi \right) \quad (8.41)$$

where the gauge covariant derivative for the fermions is

$$D_\mu \psi = \partial_\mu \psi - ig A_\mu^a T^a \psi \quad (8.42)$$

and you pick the matrices T^a depending on which representation you're in. That's it! Note that again this is the most general renormalizable action, and it depends on only a single parameter g , the gauge coupling (which is hidden in the covariant derivatives).

The classical equations of motion following from (8.41) are

$$D_\mu F^{a\mu\nu} = -g (\bar{\psi} \gamma^\nu T^a \psi) \quad (i\not{D} - m) \psi = 0 \quad (8.43)$$

The new term on the right-hand side of the gauge field equation of motion can be understood as the Noether current that you would have gotten if you had treated the gauge symmetry as a global symmetry.

Let's just discuss a few properties of this *classical* theory. Note that it appears that we have $N^2 - 1$ types of massless gauge boson, or *gluon*, one for each Lie algebra element.

Next, we note that the action above is *exactly* the action for QCD, with $N = 3$ and with a sum over a bunch of fermions called *quarks*. In this context we call the non-Abelian gauge fields “gluons”.

$$\boxed{S_{\text{QCD}}[\bar{q}, q, A] = \int d^4x \left(-\frac{1}{2} \text{tr}(F^2) + \sum_i \bar{q}_i (i\not{D} - m_i) q_i \right)} \quad (8.44)$$

where in this equation i runs over the different types of quarks (up, down, charm, strange, top, bottom), they all have different masses m_i that you can look up, $N = 3$, and each quark is in the fundamental, i.e. q_i^a , where $a \in 1 \dots 3$.

Finally, let us study some simple solutions to the Yang-Mills equation. As the quarks are in the fundamental they come in 3 types called *color*: pick a direction in this $SU(3)$ space and call it *red*. Now imagine (classically) getting one of the quarks to sit still at a point x . In that case the time-component of the current on the right-hand side of (8.43) is a delta function, and to lowest order in g , we find something that looks like

$$D_\mu F^{a\mu t} = g(t^a)_{11} \delta^{(3)}(\vec{x}) \quad (8.45)$$

The right-hand side is proportional to a particular component of the representation matrix. There are all kinds of non-linear terms on the LHS proportional to $gf^{abc}A^2$, but you can show that they do not contribute and you end up with just getting a Maxwell-type equation, which should have an inverse-square solution:

$$\partial_\mu F^{a\mu t} = g(t^a)_{11} \delta^{(3)}(\vec{x}) \quad F^{art} \sim \frac{1}{r^2} \quad (8.46)$$

This suggests that a quark should make a color electric field that falls off as $1/r^2$.

Physically, however, this is completely incorrect. In real life the gluons are not massless, and it is impossible to even *find* an isolated quark, let alone observe that it creates a color electric field that goes like $1/r^2$. Note the difference with EM: there we get more or less the right answer in the full quantum theory from just looking at the classical equations of motion. For QCD we get completely the wrong answer: apparently the classical theory is a terrible approximation to the quantum theory.

8.3 Quantizing non-Abelian gauge theories

Thus motivated, we turn now to the quantum theory. We will temporarily throw out the quarks and work out the Feynman rules for pure Yang-Mills theory, i.e.

$$S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right) \quad (8.47)$$

We turn back to a general gauge group G .

We first recall that since we have

$$F_{\mu\nu}^a = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + g A_\mu^b A_\nu^c f^{abc} \quad (8.48)$$

this theory is no longer quadratic, and thus even before we add any matter it is *already* interacting. Again, the interaction structure is completely fixed by gauge-invariance. Expanding out the action we see that we have

$$S[A] = \int d^4x \left(-\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g A_\mu^b A_\nu^c f^{abc}) (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g A_\mu^d A_\nu^e f^{ade}) \right) \quad (8.49)$$

Thus there will be an interaction vertex that couples three A 's and one that couples four of them.

We can work out the vertices through the usual techniques, which I will not go through in detail on the board as it is just bookkeeping. Note that this the first example we have seen of an interaction term that involves a derivative of the field, and this means that the Feynman rule will explicitly involve the momentum in the line connected to the vertex. I encourage you to work through the details at home (see, in particular, Exercise 2 from Week 4 for a simpler example with the same idea).

This structure is highly constrained – if you just wrote out the most general interaction there would be many other possible terms. In particular, note that it all depends on a single parameter g .

Now that we have all the vertices worked out, we should perform manipulations similar to those that we did for the Abelian case to derive the *propagator* for the gauge field. We will do precisely the same thing as we did there: we will define a gauge-fixing function

$$G_\omega^a(A) \equiv \partial_\mu A^{a\mu} - \omega^a(x) \quad (8.50)$$

Note that this is actually one gauge fixing function for each gluon. We further define a general gauge transformation by $\exp(i\alpha^a(x)t^a)$ and denote the gauge transform of A by α by A^α , whose components are:

$$(A^\alpha)_\mu^a t^a \equiv e^{i\alpha^a(x)t^a} \left(A_\mu^a(x)t^a + \frac{i}{g} \partial_\mu \right) e^{-i\alpha^a(x)t^a} \quad (8.51)$$

It is helpful to work out the infinitesimal version of this transformation, which is

$$(A^\alpha)_\mu^a = A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c = A_\mu^a + \frac{1}{g} D_\mu \alpha^a \quad (8.52)$$

where in the last term we used the formula previously derived for the covariant derivative of a field in the adjoint (the field in this case is the gauge-transformation parameter α^a). Note the difference from the Abelian case is in the term with the f^{abc} (relatedly, the fact that the gauge-transformation parameter α^a itself has a Lie algebra index). Just as before, we write:

$$1 = \int [D\alpha] \delta [G_\omega(A^\alpha)] \det \left(\frac{\delta G_\omega(A^\alpha)}{\delta \alpha} \right) \quad (8.53)$$

and plug this whole thing into the path integral to find:

$$Z = \int [DAD\alpha] \delta [G_\omega(A^\alpha)] \det \left(\frac{\delta G_\omega(A^\alpha)}{\delta \alpha} \right) \exp(iS[A]) \quad (8.54)$$

$$= g f^{abc} \left[\gamma^{\mu\nu} (k-p)^\rho + \gamma^{\nu\rho} (p-q)^\mu + \gamma^{\rho\mu} (q-k)^\nu \right]$$

$$= -ig^2 \left[f^{abe} f^{cde} (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) + f^{ace} f^{bde} (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) + f^{ade} f^{bce} (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma}) \right]$$

Figure 19: Interaction vertices for non-Abelian gauge fields

Now we will encounter a difference. Consider the operator in the functional determinant:

$$\frac{\delta G_\omega(A^\alpha)}{\delta \alpha} = \frac{\delta}{\delta \alpha} (\partial_\mu (A^\alpha)^\mu - \omega) \quad (8.55)$$

$$= \frac{\delta}{\delta \alpha} \left(\partial_\mu \left(A^\mu + \frac{1}{g} D^\mu \alpha \right) - \omega \right) \quad (8.56)$$

$$= \frac{1}{g} \partial_\mu D^\mu \quad (8.57)$$

Recall that D_μ is the covariant derivative of a field acting on the adjoint – thus there is an A in it! In other words, this time the determinant depends on A_μ , and so we cannot take it outside the functional integral.

However everything else goes on exactly as in the Abelian case – we integrate over all ω with a Gaussian weighting, etc. etc. Thus we find at the end of the day

$$Z = \int [\mathcal{D}A] \det \left(\frac{1}{g} \partial_\mu D^\mu \right) \exp \left(iS[A] - i \int d^4x \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right) \quad (8.58)$$

What about the determinant? We would like to have a useful way to represent it. Think back to two lectures ago when we first studied Grassman integrals – then we learned that if we do an integral over a Grassman variable, we get a determinant in the numerator. In other words, we are free to write

$$\det (\partial_\mu D^\mu) = \int [dc d\bar{c}] \exp \left(i \int d^4x \bar{c} (-\partial_\mu D^\mu) c \right) \quad (8.59)$$

where c^a is a Grassman field that is a scalar in spacetime but is in the *adjoint* with respect to gauge transformations. Note that its Grassman (so its anticommuting) but it is *not* a spinor. Such excitations are called *Faddeev-Popov ghosts*. I will say a lot more about them in a second.

But first let's recap: our final functional integral takes the form

$$Z = \int [\mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c}] \exp \left(i \int d^4x \left(\frac{1}{2} \text{tr}(F^2) - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \bar{c} (\partial_\mu D^\mu) c \right) \right) \quad (8.60)$$

where the action is completely gauge fixed, at the costs of introducing mysterious new particles called *ghosts*.

So first the good news – the quadratic part of the action is just like $d(G)$ free Maxwell actions, and the new gauge-fixing term is diagonal in this space. So we can simply repeat the previous analysis to find the non-Abelian gauge field propagator:

$$D_{\mu\nu}^{ab}(p) = -\frac{i}{p^2} \left(\eta_{\mu\nu} - (1 - \xi) \frac{p_\mu p_\nu}{p^2} \right) \delta^{ab} \quad (8.61)$$

exactly the same as the Abelian one, except with an extra δ^{ab} making it diagonal in color space. In this case the choice of gauge parameter $\xi = 1$ is called Feynman-t'Hooft gauge.

Now we turn to the ghosts. Writing out their Lagrangian explicitly we find

$$S_{\text{ghost}}[c, \bar{c}] = \int d^4x \bar{c}^a (-\partial^2 \delta^{ac} - g \partial^\mu f^{abc} A_\mu^b) c^c \quad (8.62)$$

This is a completely normal Lagrangian – we can easily work out the ghost propagator and the interaction vertex. The Feynman rules are shown in Figure 21.

The ghost propagator is

$$D_{\text{ghost}}^{ab}(p) = \frac{i}{p^2} \delta^{ab} \quad (8.63)$$

$$i \longrightarrow j = \frac{i}{\not{k} - m} \delta_{ij}$$

Figure 22: Propagator for fermion transforming in fundamental

Apparently to calculate things we have to include these ghosts. Let me stress that they actually quantitatively change things: for example, the vacuum polarization diagram for a gluon receives a contribution from a ghost loop, which is calculable **Figure**.

Having said that, this is the *complete set of Feynman rules*. Now you can calculate anything you like in perturbation theory: gluon-gluon scattering, etc. Unfortunately the course is almost over, so we won't really get to, though I hope to say some qualitative things.

Now let's return to these ghosts. Here are some questions you may have:

1. *Q. Can I make them in a particle physics experiment?*

A. No, they are just fictitious objects that we made up to represent a functional determinant. Of course they run in loops and can affect Feynman diagrams, but the construction does not permit an "external ghost line", so you need never worry about making a ghost at the LHC.

2. *Q. Shouldn't I be worried that they are scalar particles that are anticommuting? You mumbled something about spin and statistics earlier.*

A. Indeed, they violate the spin-statistics theorem. This is okay, because as I said earlier, they can never be external particles.

3. *Q. What are they the ghosts of?*

A. Interestingly enough, this question has an answer. Recall that the gauge field has 4 components but only 2 actual physical polarizations, meaning that 2 polarizations are spurious. Now imagine a loop – naively these 2 spurious components want to run in the loop, but we shouldn't let them (remember, they are unphysical). One way to interpret the ghosts is that due to the anticommuting minus signs, they *cancel* the two spurious polarizations. This is the reason the ghosts exist.

Finally, we can put back the quarks. If we do this the quark part of the action

$$S_{\text{quark}}[\bar{q}, q, A] = \int d^4x \bar{q}^i (i\gamma^\mu (\partial_\mu - igA_\mu^a t^a)_{ij} q^j) \quad (8.64)$$

where I have temporarily put back the color indices $i \in 1, \dots, N$ on the quarks. From here, we can read off the quark propagator and the interaction of the quark with the gauge field:

- The propagator is the usual one, but is now diagonal in color space: $D_F^{ij}(p) = \frac{i\delta^{ij}}{\not{p} - m}$
- The interaction vertex is $-ig\gamma^\mu t_{ij}^a$. Note the representation matrix plays the role of tying together all of the indices.

From these we can now calculate anything that we like.

8.4 Qualitative discussion: Non-Abelian gauge theory at long distances

These are the Feynman rules – we can now use them to calculate things. In practice, we are almost out of time, so we will not do much quantitatively, but I will try to vaguely argue for what can happen. This ends the precise part of the course, from now on I will be hand-wavy.

Let's remember when we first calculated the one-loop correction to the amplitude for 2-2 scattering in $\lambda\phi^4$ theory. We calculated the diagrams in Section 4 and found that at high energies the amplitude as a function of s (and t, u). At large $s \gg m$ it went like

$$i\mathcal{M} = -i\lambda - i\frac{\lambda^2}{32\pi^2} \left(\log\left(\frac{s}{s_0}\right) + \dots \right) \quad (8.65)$$

where s_0 was a renormalization point and I've not written down the dependence on t and u (but it is similar). Now from staring at this, you can conclude that the interaction λ gets *stronger* at high energies, as increasing s acts in the same direction as increasing λ . Also note this depends crucially on the *sign* of the prefactor of the one-loop term: if it had the other sign, then increasing s would have decreased the interaction strength. You can do a similar calculation in QED and arrive at a similar answer, QED also gets strongly interacting in the UV.

Now let's imagine doing a similar calculation in non-Abelian gauge theory: the goal is to understand whether it gets stronger or weaker as we go to high energies. We need to pick some interaction process: there are many different processes that we could pick, they should all give the same result because they are all controlled by the same number g . So we pick the quark-quark-gluon coupling and calculate. We look at the diagrams. There is only one at tree level, but there are several at one loop.

So now we calculate them all (see Chapter 16.5 of Peskin), then we get a result that looks like (very very schematically)

$$i\mathcal{M} = -ig + i\frac{g^3}{(4\pi)^2} \left(\frac{11}{3}N - \frac{2}{3}n_q \right) \left(\log\left(\frac{E}{E_0}\right) + \dots \right) \quad (8.66)$$

where n_q is the number of quark fields, and where E is the energy flowing into the vertex. I say this is schematic because I am being deliberately vague about what I mean by “energy flowing into the vertex”: it is clear that there are several different energies you could pick. So actually the correct way to parametrize this energy is to *not* do what I described, but to instead compute the *beta function* of the coupling g (again, to be done in the course on renormalization). However this gives you a reasonable idea of what is happening.

The crucial thing to note here is the *sign*! For a small enough number of quark fields n_q , it looks like this thing has the opposite sign as the ϕ^4 theory, and so this means that the theory becomes *weakly coupled* – in fact, a free theory – in the UV. This property is called *asymptotic freedom*: in the context of QCD, it was first discovered by Gross, Politzer and Wilczek, for which they won the Nobel Prize in 2006. To appreciate its importance, we should think not in the UV, but in the IR – if the interaction is weak in the UV, then it seems to be getting stronger and stronger at low energies. So what happens?

There are a few possibilities – one thing that could happen is that the theory ends up getting Higgsed. This happens for the $SU(2) \times U(1)$ part of the Standard Model; you have studied this in that course, so I will discuss it no further here. I stress that you can often still understand that within perturbation theory – you just expand around the new vacuum.

But let us assume this does not happen, and the interaction just gets stronger and stronger. Now you have to take my work on what happens – from a renormalization group analysis it turns out that the strength of the interaction diverges and goes to infinity at a scale that is

$$\Lambda_{QCD} = E_0 \exp\left(-\frac{8\pi^2}{b_0 g^2}\right) \quad (8.67)$$

where E_0 is the renormalization point (recall our definition of the renormalized coupling g depends on this choice of renormalization point), and $b_0 = (\frac{11}{3}N - \frac{2}{3}n_q)$. This is the *strong-coupling scale* of non-Abelian gauge theory: at this scale we lose all hope of doing perturbation theory. This is why our classical intuition does not work well for non-Abelian gauge theory. It is thus scale at which bound states are expected to form – a bound state is a very non-perturbative thing.

So now we have a picture of what is happening for QCD. In the UV, we have a theory of quarks and gluons. They interact and run around in a manner that can be understood in perturbation theory. However, if you go to long distances, the theory becomes more and more strongly interacting. Eventually we don't understand perturbation theory any more, but it *looks* like all the quarks and gluons are bound up into bound states that are all singlets under the gauge transformations. (If they were not singlets, they would interact with the gauge field, which is strongly interacting, and that would be bad – therefore all bound states are singlets). This phenomenon – where in the IR you have only gauge-singlet bound states, and no massless gauge bosons – is called *confinement*.

How do we make gauge-singlets out of quarks? Schematically, there are two ways to combine fundamental degrees of freedom into gauge singlets. We can imagine having $\bar{q}q$ – these are called *mesons*, and examples are the pions. Or we could have $\epsilon^{ijk}q_iq_jq_k$, which are gauge singlets since the determinant of an $SU(3)$ matrix is 1. These are called *baryons*, and two examples that we know and love are the proton and neutron. I stress that these expressions are schematic things designed to capture the quantum numbers – the fields that create mesons are not *actually* simple products of the quark fields, but rather products of the quark fields surrounded by a gluon cloud.

Note that we expect all the masses of these bound states to be around Λ_{QCD} , which is roughly 200 MeV. Indeed the proton and neutron have masses around 938 MeV, while the pions have masses of around 135 MeV (this relative low-ness has to do with chiral symmetry breaking, which I will not discuss).

8.4.1 What are the gauge fields doing in a confined phase?

Now I will go even more qualitative. I would like to try to explain what the gauge fields are doing in the confined phase.

Let's first get oriented by going back to normal $U(1)$ gauge theory, i.e. electromagnetism. Imagine having an electron and a positron separated by a distance R . The positron makes electric field lines that come out and go into the electron – because EM is weakly coupled, we can solve for their shape, and it is an elementary problem in EM to show that the Coulomb energy from the interaction between the two particles goes like

$$E(R) \sim \frac{e^2}{R} \tag{8.68}$$

Now this is *not* what happens in non-Abelian gauge theory. Imagine pinning down a quark and an anti-quark separated by a distance R . They make *color* electric fields – but what we *expect* to happen is that the color electric fields do not spread out in all directions like they do in free EM. Instead they remain focused into a tight tube called a *flux* tube that attaches the quark to the antiquark. This tube actually has a tension, and so the energy is

$$E(R) \sim \alpha' R \tag{8.69}$$

where I have called the tension of the flux tube α' for some reason. But now note that you cannot separate two quarks – if you try to pull them apart, then you pull on this flux tube and this costs infinite energy. It is like the two quarks are attached by a string. This is the idea of confinement. In this picture you can imagine that a meson is a quark and an antiquark attached by a bit of string – the different vibrational modes of the bound state are the different mesons that exist in real life.

This is pretty much as far as we will go. There are a few things you could try to do now: one promising idea was to try to take the flux tube as the starting point: in other words, try to build a theory out of quantizing the dynamics of flux tubes. The problem with this approach is that it *always* resulted in a spin-2 massless particle, which is really annoying because if you look at the QCD spectrum there is no such particle.

Of course eventually it was realized that this theory of strings was really solving an entirely different problem, and this is how string theory as a theory of quantum gravity was discovered. This spin-2 particle is the graviton.

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