Introduction to String Theory

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– based on notes by Iñaki García Etxebarria –

Contents

Preliminaries

There are a number of good textbooks on String Theory. The things that we have discussed during the lectures are explained well in all of them, they mostly differ in their treatment of less introductory topics.

- *Superstring Theory*, by Green, Schwarz and Witten. This dates back to 1987, so it does not contain any of the modern developments, but what is there is superbly explained. I am mostly following (parts of) this book.

- *String Theory*, by Polchinski. This is a somewhat more modern approach, and it has become the standard reference in the field. This is a good place to take a look for an alternative viewpoint on quantization, where the fact that we are dealing with a CFT in two dimensions takes a much more prominent role.

- *Basic Concepts of String Theory*, by Blumenhagen, Lüst and Theisen is also quite good. It is particularly noteworthy in that it is fairly thorough in its derivations: if you find one of my arguments too quick, it might be helpful to look here to fill in the details, with all signs and factors in place.

- *D-Branes*, by Johnson also has a good introduction to the quantization of the classical string. I particularly like its section on more advanced material: if you are curious about what string theorists have actually been up to during the last thirty years, after what I explained during the lectures was understood, this might be a good and fairly accessible starting point.

- *A First Course in String Theory*, by Zwiebach. This is a very clear and lucid text, which includes a careful discussion of a lot of background material that the previous references often take for granted.

You can also easily find excellent material on string theory online, a particularly good account can be found in David Tong's lecture notes:

- `http://www.damtp.cam.ac.uk/user/tong/string.html`.

If you have any questions, or you find any mistakes in these notes, please send me an email to andreas.braun@durham.ac.uk.

§1 The relativistic point particle

§1.1 *The classical particle as a theory of gravity in one dimension*

We will start by studying, from a perhaps somewhat unconventional viewpoint, the case of a massive particle propagating in flat D-dimensional space. The relativistic action for a particle of mass m, in units where $c = 1$, is given by

$$S = -m \int dt \sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}\,. \tag{1.1}$$

where $\dot{\vec{x}} \cdot \dot{\vec{x}} := \sum_{i=1}^{D-1} \dot{x}_i^2$.

> ✎ **Exercise 1.1.** *Show that the canonical momenta and energy for this theory are:*
>
> $$p_i = \frac{m\dot{\vec{x}}}{\sqrt{1 - \dot{\vec{x}} \cdot \dot{\vec{x}}}} \qquad ; \qquad E = \sqrt{m^2 + \vec{p} \cdot \vec{p}}\,. \tag{1.2}$$

This expression looks fairly asymmetric between space and time, an equivalent form that better shows the underlying Poincaré invariance of the theory is given by

$$S = -m \int ds = -m \int d\tau \sqrt{-\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau}} \tag{1.3}$$

where $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, \ldots, +1)$ is the Minkowski metric in D-dimensions, and ds is the length element along the string induced by the embedding. Here τ is an arbitrary parameter along the worldline of the particle. This form of the action is clearly invariant under Poincaré transformations $x^\mu \to \Lambda^\mu_\nu x^\nu + c^\mu$, with $\Lambda^\mu_\rho \Lambda^\nu_\sigma \eta_{\mu\nu} = \eta_{\rho\sigma}$.

It is also easy to verify that the action does not depend on the choice of τ: if we introduce a new parametrization $\tilde{\tau}(\tau)$ along the worldline we have $d\tilde{\tau} = \frac{d\tilde{\tau}}{d\tau} d\tau$ for the measure and

$$\frac{dx^\mu}{d\tau} = \frac{dx^\mu}{d\tilde{\tau}} \frac{d\tilde{\tau}}{d\tau} \tag{1.4}$$

for the velocities, so (1.3) is indeed invariant under reparametrizations. The fact that the action is invariant under reparametrizations allows us to choose coordinates where $\tau = x^0 := t$, and in this way we go back to (1.1).

So far we have talked about a particle moving in D-dimensions, but a different viewpoint is useful when thinking about generalizing to string theory. The $x^\mu(\tau)$ are functions parametrizing abstract embeddings of a 1-dimensional object into D-dimensions, but we can equivalently think of them as fields in a one-dimensional theory. Then (1.3) gives us a rather complicated action for these fields in this one-dimensional field theory. In particular, the action includes a square root term, which make quantization difficult.

Interestingly, we can write an action which is classically equivalent, but which includes no square roots, by coupling to one-dimensional gravity. Here it is:

$$\begin{aligned} S &= \frac{1}{2}\int d\tau \left(e^{-1}\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} - em^2 \right) \\ &= -\frac{1}{2}\int d\tau\sqrt{-g_{\tau\tau}}\left(g^{\tau\tau}\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + m^2 \right) \end{aligned} \tag{1.5}$$

Here $e := \sqrt{-g_{\tau\tau}}$ is an "einbein", and $g_{\tau\tau}$ is the metric along the worldline. In this formulation we treat e (or equivalently, $g_{\tau\tau}$) as dynamical fields. Note that one interesting feature of (1.5) is that, contrary to (1.3), taking the $m \to 0$ limit poses no particular problem.

The equation of motion for e is

$$\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + e^2m^2 = 0 \tag{1.6}$$

which can be used to solve for e in terms of the embedding:

$$e^2 = -\frac{1}{m^2}\eta_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau}\,. \tag{1.7}$$

If we plug this result into (1.5) we recover (1.3), showing that the two actions are classically equivalent. Note that, using the identification $e = \sqrt{-g_{\tau\tau}}$, (1.7) provides an on-shell relation between the one-dimensional metric that we have introduced and the metric induced by the embedding of the worldline in spacetime.

The punchline of all this is rather interesting: we have just seen that one can reformulate (classically at least) the theory of a relativistic particle moving in D dimensions as a quantum field theory with gravity in one dimension, where the embedding coordinates are one dimensional fields.

As is probably familiar from your studies of general relativity, the gravitational action (1.5) is invariant under infinitesimal reparametrizations $\tau \to \tau + \xi(\tau)$:

$$\delta x^\mu = \xi\frac{\partial x^\mu}{\partial\tau} \tag{1.8a}$$

$$\delta e = \frac{\partial}{\partial\tau}(\xi e)\,. \tag{1.8b}$$

✎ **Exercise 1.2.** *Show that the action (1.5) is invariant under this reparametrization, up to boundary terms at* $\tau = \pm\infty$.

We can use this invariance under reparametrizations to fix a convenient form for e, for instance by choosing e to be a convenient constant, for instance $e = 1/m$. But it is important to keep in mind that (1.7) still needs to be imposed, as otherwise we are introducing new degrees of freedom not present in the original system. That is, instead of considering (1.5) we could equivalently consider the action

$$S = \frac{m}{2} \int d\tau \left(\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} - 1 \right) \tag{1.9}$$

with solutions subject to the additional constraint $\eta_{\mu\nu} \frac{dx^\mu}{d\tau} \frac{dx^\nu}{d\tau} + 1 = 0$. Note that this form of the action is much easier to treat: since $\eta_{\mu\nu}$ is diagonal we simply have D decoupled massive fields in one dimension, which we can now quantize rather straightforwardly.

§1.2 *The interacting classical particle*

Let me comment briefly on how particle interactions look from this point of view. A freely propagating particle has a worldline which is topologically the real line $\mathbb{R}$. Adding interactions corresponds to defining the theory on more complicated (singular) one dimensional manifolds, built out of interaction vertices, such as those in figure 1. Note that defining the one-dimensional theory on each interaction vertex requires the specification of an additional piece of data, the coupling constant for the corresponding interaction.

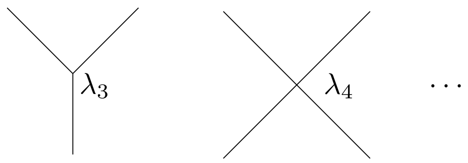


Figure 1: Interaction diagrams for the one dimensional theory.

One can push this viewpoint slightly further, but it soon becomes rather clumsy to deal with. Interestingly, in the case of string theory some of these difficulties are ameliorated, and we can make a lot of progress staying within the perspective of a gravitational theory on the worldsheet, the two-dimensional generalization of the worldline we have been considering here. In coming sections, when we discuss how to quantize the string, what we will quantize is the two-dimensional gravity theory on the string worldsheet.

It is important to emphasize that in the case of interacting particles in D dimensions this is certainly *not* what we usually do: instead of trying to quantize the 1d theory defined above we rather study quantum field theory in D dimensions, a very different problem. Much of the mystery and power of string theory lays on the tension between the different formulations that we have: from the point of view of the worldsheet spacetime is a secondary notion, but ultimately we will be describing particles propagating on D dimensions, where the right description seems to be D dimensional QFT. While we do not yet understand what is the right fundamental formulation that unifies the worldsheet and spacetime viewpoints (or even if one exists at all!) in practice assuming that one can combine both perspectives, using each whenever applicable, does get us very far.

§2 The classical bosonic string

§2.1 The Nambu-Goto and Polyakov actions

I would now like to generalize the previous discussion to two dimensional objects (i.e. "strings") propagating in flat D-dimensional Minkowski spacetime. The natural two-dimensional generalization of (1.3) is

$$S_{NG} = -T \int_\Sigma d(\text{area}) = -T \int_\Sigma d^2\boldsymbol{\sigma}\sqrt{-\det(\gamma)} \tag{2.1}$$

where Σ is the two-dimensional worldsheet that we want to understand, T is its tension (more on this below), $\boldsymbol{\sigma} := (\sigma^0, \sigma^1)$ is some parametrization of Σ, and finally

$$\gamma_{ab} = \frac{dX^\mu(\boldsymbol{\sigma})}{d\sigma^a}\frac{dX^\nu(\boldsymbol{\sigma})}{d\sigma^b}\eta_{\mu\nu} \tag{2.2}$$

is the metric induced on Σ by the embedding on spacetime (with the embedding parametrized by the maps $X^\mu\colon \Sigma \to \mathbb{R}^{1,D-1}$).

This form of the action is known as the *Nambu-Goto action*. We will be interested in quantizing this theory, but the square root makes this rather awkward. Similarly to what we did in the case of the point particle, we can switch to a classically equivalent description that is easier to quantize by coupling to two-dimensional gravity. This description of the dynamics of the classical string is known as the *Polyakov action*:

$$S_P = -\frac{T}{2}\int_\Sigma d^2\boldsymbol{\sigma}\sqrt{h}h^{ab}(\boldsymbol{\sigma})\partial_a X^\mu \partial_b X^\nu \eta_{\mu\nu} \tag{2.3}$$

Here $h_{ab}(\boldsymbol{\sigma})$ is a metric on the worldsheet, and $h^{ab}(\boldsymbol{\sigma})$ is its inverse. (I will be using conventions where latin letters towards the beginning of the alphabet, such as a, b denote worldsheet indices, and greek letters such as μ, ν denote spacetime indices). We also define $\sqrt{h} := \sqrt{|\det(h_{ab})|}$ for convenience, and use the notation $\partial_a X^\mu := \frac{\partial X^\mu}{\partial \sigma^a}$.

We can show that S_P and S_{NG} are classically equivalent easily. The (inverse) worldsheet metric h^{ab} appears without derivatives in the action, so its Euler-Lagrange equation of motion is simply

$$\frac{\delta S_P}{\delta h^{ab}} = -\frac{T}{2}\sqrt{h}T_{ab} = 0 \tag{2.4}$$

where we have introduced the energy-momentum tensor T_{ab}

$$T_{ab} := -\frac{2}{T}\frac{1}{\sqrt{h}}\frac{\delta S_P}{\delta h^{ab}}\,. \tag{2.5}$$

✎ **Exercise 2.1.** *Show that*

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2} h_{ab} h^{cd} \partial_c X^\mu \partial_d X_\mu \,. \tag{2.6}$$

It is consistent to restrict to worldsheet metrics with $\sqrt{h} \neq 0$, so we can write the h equation of motion as $T_{ab} = 0$. Note that this equation can be written in terms of the induced worldsheet metric (2.2) as

$$\gamma_{ab} = \frac{1}{2} \left(h^{cd} \gamma_{cd} \right) h_{ab} \,. \tag{2.7}$$

This implies, in particular, that on-shell (that is, classically) the worldsheet and induced metrics are related. Taking the determinant of this equation we find $\det(\gamma) = \frac{1}{4} \left(h^{cd} \gamma_{cd} \right)^2 \det(h)$, which immediately gives S_P when subtituted into S_{NG}.

While the two actions are classically equivalent, it will be much easier to quantize the Polyakov action S_P, so from now on we will choose the quantum theory to be the one with action S_P.

§2.2 Symmetries and gauge fixing

Let us now discuss more systematically the symmetries of the Polyakov action.

1. **Reparametrization invariance.** We have a theory of gravity in two dimensions, so in particular the theory is invariant under reparametrizations $\boldsymbol{\sigma} \to \boldsymbol{\sigma}'(\boldsymbol{\sigma})$ of the worldsheet. We have

$$X'^\mu(\boldsymbol{\sigma}') = X^\mu(\boldsymbol{\sigma}) \,, \tag{2.8a}$$

$$\frac{\partial \sigma'^c}{\partial \sigma^a} \frac{\partial \sigma'^d}{\partial \sigma^b} h'_{cd}(\boldsymbol{\sigma}') = h_{ab}(\boldsymbol{\sigma}) \,, \tag{2.8b}$$

or equivalently, in terms of the infinitesimal transformation $\sigma'^a = \sigma^a + \xi^a(\boldsymbol{\sigma})$

$$\delta X^\mu = \xi^a \partial_a X^\mu \tag{2.9a}$$

$$\delta h^{ab} = \xi^c \partial_c h^{ab} - h^{cb} \partial_c \xi^a - h^{ac} \partial_c \xi^b \tag{2.9b}$$

$$\delta(\sqrt{h}) = \partial_a (\xi^a \sqrt{h}) \,. \tag{2.9c}$$

2. **Weyl scaling.** The Polyakov action is also invariant under Weyl rescalings of the metric:

$$X'^\mu(\boldsymbol{\sigma}) = X^\mu(\boldsymbol{\sigma}) \tag{2.10a}$$

$$h'^{ab}(\boldsymbol{\sigma}) = e^{\Theta(\boldsymbol{\sigma})} h^{ab}(\boldsymbol{\sigma}) \tag{2.10b}$$

where $\Theta(\sigma)$ is an arbitrary function on the worldsheet. The infinitesimal form of these transformations is

$$\delta X^\mu = 0\,, \tag{2.11a}$$
$$\delta h^{ab} = \Theta h^{ab}\,. \tag{2.11b}$$

✎ **Exercise 2.2.** *Show that Weyl invariance of S_P implies $h_{ab}T^{ab} = 0$, without using the equations of motion.*

3. **Poincaré invariance.** Finally, the theory is also invariant under Poincaré transformations

$$X'^\mu = \Lambda^\mu{}_\nu X^\nu + b^\mu \tag{2.12a}$$
$$h'^{ab} = h^{ab} \tag{2.12b}$$

with $\Lambda^\mu{}_\nu$ generators of the Lorentz group, satisfying $\Lambda^\mu{}_\nu \Lambda^\rho{}_\sigma \eta_{\mu\rho} = \eta_{\nu\sigma}$. Both $\Lambda^\mu{}_\nu$ and b^μ are independent of $\boldsymbol{\sigma}$. Infinitesimally:

$$\delta X^\mu = a^\mu{}_\nu X^\nu + b^\mu \tag{2.13a}$$
$$\delta h^{ab} = 0\,, \tag{2.13b}$$

with $a_{\mu\nu} = \eta_{\mu\rho} a^\rho{}_\nu$ antisymmetric.

Note that the last symmetry is of a somewhat different nature from the previous two: from the point of view of the worldsheet it is a *global* symmetry, which will have associated Noether currents. These conserved currents will play an important role later when we try to interpret the oscillation modes of the string in terms of spacetime particles. On the other hand, the first two symmetries are *local* (or gauge) symmetries, and they lead to constraints.

It is useful to use the gauge symmetries to transform h_{ab} into a more convenient form. In two dimensions the metric has three independent components. Reparametrizations are generated by two independent functions, and Weyl rescalings by a third function, so we expect to be able to locally set h_{ab} to any convenient form that we like by a suitable choice of gauge. We will choose

$$h_{ab} = \eta_{ab} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.14}$$

With this choice, the Polyakov action (2.3) simplifies to

$$S = -\frac{T}{2}\int_\Sigma d^2\boldsymbol{\sigma}\, \eta^{ab}\partial_a X^\mu \partial_b X_\mu\,. \tag{2.15}$$

It is important to remember that in order to reproduce the dynamics before gauge fixing we still need to impose the $T_{ab} = 0$ equation of motion for h, we will come back to this point in §2.6 below.

In this gauge the X^μ are free fields, obeying the one-dimensional wave equation

$$\left(\partial_0^2 - \partial_1^2\right) X^\mu(\boldsymbol{\sigma}) = 0\,. \tag{2.16}$$

The general solution of this differential equation, due to D'Alembert, is well known. Introduce $\sigma^\pm := \sigma^0 \pm \sigma^1$. Then

$$X^\mu(\boldsymbol{\sigma}) = X^\mu_R(\sigma^-) + X^\mu_L(\sigma^+)\,. \tag{2.17}$$

It will often prove very useful to work in the $\sigma^\pm$ coordinate system, so let us briefly describe how things work in these coordinates. We have

$$\eta_{+-} = \eta_{-+} = -\frac{1}{2} \qquad ; \qquad \eta_{++} = \eta_{--} = 0 \tag{2.18}$$

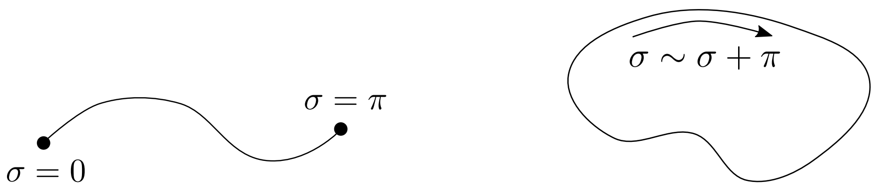
and therefore

$$\eta^{+-} = \eta^{-+} = -2 \qquad ; \qquad \eta^{++} = \eta^{--} = 0\,. \tag{2.19}$$

Raising and lowering indices is done by the metric, so $U^+ = -2U_-$ and $U^- = -2U_+$ for any vector U. Partial derivatives are given by $\partial_\pm = \frac{1}{2}(\partial_0 \pm \partial_1)$.

§2.3 *Boundary conditions: open and closed strings*

So far we have only discussed what happens locally in the two dimensional theory living on the string. We now discuss how the global structure of the string affects the dynamics. We will consider two spatial topologies for the string, either *open* or *closed*, as shown in figure 2.



(a) An open string. (b) A closed string.

Figure 2: Constant τ slices of open and closed strings.

More precisely, we will take $\tau := \sigma^0$ to be the "time" coordinate in the string, and $\sigma := \sigma^1$ as the spatial coordinate, and introduce the notation

$$\dot{X}^\mu := \frac{\partial X^\mu}{\partial \tau} \qquad \text{and} \qquad X'^\mu := \frac{\partial X^\mu}{\partial \sigma}\,. \tag{2.20}$$

For both open and closed strings we take $\tau \in \mathbb{R}$. The freely propagating open string is described by a worldsheet with the topology of a strip, with two boundaries, which we choose to put at $\sigma = 0$ and $\sigma = \pi$. The freely propagating closed string is described by a worldsheet with the topology of an infinite cylinder. We choose conventions so that the periodicity is $\sigma \sim \sigma + \pi$.

Closed strings

In the case of the closed string we take $X^\mu(\sigma) = X^\mu(\sigma+\pi)$, consistent with the periodicity $\sigma \sim \sigma+\pi$. (This is not the only possibility, for instance if a spacetime direction is periodic we can have winding modes, see below.) In this case the general solution (2.17) admits a mode expansion of the form

$$X^\mu_R(\sigma^-) = \frac{1}{2}(x^\mu + c^\mu) + \frac{1}{2}\ell^2 p^\mu \sigma^- + \frac{i}{2}\ell \sum_{n\neq 0} \frac{1}{n}\alpha^\mu_n e^{-2in\sigma^-} \ , \tag{2.21a}$$

$$X^\mu_L(\sigma^+) = \frac{1}{2}(x^\mu - c^\mu) + \frac{1}{2}\ell^2 p^\mu \sigma^+ + \frac{i}{2}\ell \sum_{n\neq 0} \frac{1}{n}\tilde{\alpha}^\mu_n e^{-2in\sigma^+} \ . \tag{2.21b}$$

We have introduced a number of constants to parametrize the solution. It is convenient, and will not affect our ensuing discussion to set $c^\mu = 0$, so we will do so henceforth. The constant x^μ can be interpreted as the position of the centre of mass of the string in spacetime. As we will see below, p^μ can be given the interpretation of its spacetime momentum. The α^μ_n and $\tilde{\alpha}^\mu_n$ are the coefficients measuring the excitation of the n-th right and left moving modes. We require that the X^μ are real, so this requires

$$(\alpha^\mu_n)^* = \alpha^\mu_{-n} \qquad ; \qquad (\tilde{\alpha}^\mu_n) = \tilde{\alpha}^\mu_{-n} \, . \tag{2.22}$$

Finally, ℓ is a dimensionful parameter. By convention we set

$$\ell := \frac{1}{\sqrt{\pi T}} \tag{2.23}$$

and often will also introduce another related dimensionful parameter

$$\alpha' := \frac{\ell^2}{2} = \frac{1}{2\pi T} \, . \tag{2.24}$$

Open strings

Open strings admit a similar mode expansion. Consider the variation of the action (2.15). The condition $\delta S = 0$ has a bulk contribution that vanishes due to the equations of motion, and a remaining boundary contribution of the form

$$\delta S = -T \int d\tau \left[X'_\mu \delta X^\mu \Big|_{\sigma=\pi} - X'_\mu \delta X^\mu \Big|_{\sigma=0} \right] = 0 \, . \tag{2.25}$$

✎ **Exercise 2.3.** *Show that* (2.25) *is correct. You can assume that* $\delta X^\mu(\pm\infty, \sigma) = 0$.

There are essentially two ways that (2.25) can vanish at each boundary: either we impose that $\delta X^\mu = 0$ at the boundary, or we impose that $X'_\mu = 0$. The first condition is known as a *Dirichlet* boundary condition. It leads to very interesting physics, but it breaks translations in the X^μ directions, so for now we focus on the other boundary condition, $X'_\mu = 0$. This is known as a *Neumann* boundary condition.

In terms of the oscillator expansion, the Neumann boundary condition relates left and right movers, and requires the general solution to be of the form

$$X^\mu(\tau, \sigma) = x^\mu + \ell^2 p^\mu \tau + i\ell \sum_{n\neq 0} \frac{1}{n} \alpha_n^\mu e^{-in\tau} \cos(n\sigma)\,. \tag{2.26}$$

§2.4 *Noether charges for the Poincaré symmetry*

Recall that the Poincaré symmetries (2.13) are global symmetries from the point of view of the worldsheet, so they should have associated Noether charges, which will be conserved. Recall that for a general symmetry acting on a field ϕ

$$\phi(\boldsymbol{\sigma}) \to \phi(\boldsymbol{\sigma}) + \epsilon\delta\phi(\boldsymbol{\sigma}) + \mathcal{O}(\epsilon^2) \tag{2.27}$$

the associated Noether current is given by

$$J^a = \delta\phi \frac{\delta S}{\delta(\partial_a \phi)}\,. \tag{2.28}$$

This current is conserved, $\partial_a J^a = 0$, and thus leads to a conserved charge

$$Q = \int d\sigma\, J^0 \tag{2.29}$$

satisfying $\partial_\tau Q = 0$ (assuming suitable behaviour at the boundary).

Let us consider the case of translations in the target space: $X^\mu \to X^\mu + b^\mu$ with b^μ constant. We obtain in this way D conserved currents

$$(P^\mu)^a = -T\partial^a X^\mu \tag{2.30}$$

with associated conserved charges:

$$P^\mu = \int_0^\pi d\sigma (P^\mu)^0 = -\int_0^\pi d\sigma\, T\partial^0 X^\mu = \int_0^\pi d\sigma\, T\partial_\tau X^\mu = \pi T\ell^2 p^\mu = p^\mu \tag{2.31}$$

justifying our claim above that p^μ could be interpreted as the spacetime momentum for the string.

✎ **Exercise 2.4.** *Show that the Neumann boundary conditions $X'_\mu = 0$ ensure that no momentum flows out of the end of the open string.*

✎ **Exercise 2.5.** *Show that the conserved charge for Lorentz rotations is*

$$J^{\mu\nu} = l^{\mu\nu} + E^{\mu\nu} \tag{2.32}$$

with $l^{\mu\nu} = x^\mu p^\nu - x^\nu p^\mu$ and

$$E^{\mu\nu} = -i \sum_{n=1}^{\infty} \frac{1}{n}(\alpha^\mu_{-n}\alpha^\nu_n - \alpha^\nu_{-n}\alpha^\mu_n)\,. \tag{2.33}$$

§2.5 Poisson brackets

We will soon quantize the theory on the string worldsheet, and we will do so by canonical quantization. In order to do this, we need to know the canonical structure of the classical theory, or in other words the structure of Poisson brackets. Since we have D copies of a free theory in two dimensions this is fairly straightforward: the canonical momenta for the X^μ are given by

$$\Pi_\mu = \frac{\delta S}{\delta \dot{X}^\mu} = T\dot{X}_\mu \tag{2.34}$$

and thus from the fundamental Poisson brackets $\{X^\mu(\sigma), \Pi_\nu(\sigma')\}_{\text{P.B.}} = \delta(\sigma-\sigma')\delta^\mu_\nu$ we learn

$$\{X^\mu(\sigma), \dot{X}^\nu(\sigma')\}_{\text{P.B.}} = T^{-1}\delta(\sigma-\sigma')\eta^{\mu\nu}\,, \tag{2.35a}$$

and

$$\{X^\mu(\sigma), X^\nu(\sigma')\}_{\text{P.B.}} = \{\dot{X}^\mu(\sigma), \dot{X}^\nu(\sigma')\}_{\text{P.B.}} = 0\,. \tag{2.35b}$$

We will also need the Poisson brackets between the oscillators in the oscillator expansions (2.21) and (2.26). Let us do the case of the closed string, for concreteness (the open string works very similarly). As you can easily verify,

$$\tilde{\alpha}^\mu_m = \frac{1}{2\pi\ell}\int_0^\pi d\sigma e^{2im\sigma}\left[\dot{X}^\mu(0,\sigma) + \frac{2m}{i}X^\mu(0,\sigma)\right] \tag{2.36}$$

and similarly

$$\alpha^\mu_m = \frac{1}{2\pi\ell}\int_0^\pi d\sigma e^{-2im\sigma}\left[\dot{X}^\mu(0,\sigma) + \frac{2m}{i}X^\mu(0,\sigma)\right]\,. \tag{2.37}$$

Since the Poisson bracket is linear, the Poisson brackets (2.35) then lead to

$$\{\alpha^\mu_m, \alpha^\nu_n\}_{\text{P.B.}} = \{\tilde{\alpha}^\mu_m, \tilde{\alpha}^\nu_n\}_{\text{P.B.}} = -im\delta_{m+n}\eta^{\mu\nu} \tag{2.38a}$$
$$\{x^\nu, p^\mu\}_{\text{P.B.}} = \eta^{\mu\nu} \tag{2.38b}$$

with all other Poisson brackets vanishing.

§2.6 Constraints

We still need to impose the h equation of motion, or equivalently $T_{ab} = 0$, where

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu - \frac{1}{2}\eta_{ab}\eta^{cd}\partial_c X^\mu \partial_d X_\mu \,. \tag{2.39}$$

It is convenient to introduce some notation in order to write the equations more concisely. Define $A \cdot B \coloneqq A^\mu B_\mu$ and $A^2 \coloneqq A \cdot A$. We have

$$T_{00} = T_{11} = \frac{1}{2}\left(\dot{X}^2 + X'^2\right) = 0\,, \tag{2.40a}$$
$$T_{01} = T_{10} = \dot{X} \cdot X' = 0\,. \tag{2.40b}$$

Note that $T_{00} = T_{11}$ follows from Weyl invariance since (as you were asked to show in exercise 2.2) Weyl invariance implies $\eta^{ab}T_{ab} = -T_{00} + T_{11} = 0$. These equations can be alternatively written as $(\dot{X} \pm X')^2 = 0$.

In terms of the $+-$ coordinates we have that the fact that the energy-momentum tensor is symmetric, $T_{+-} = T_{-+}$, together with the tracelessness condition $-2(T_{-+} + T_{+-}) = 0$ implies $T_{+-} = T_{-+} = 0$. The non-trivial equations are

$$T_{++} = \partial_+ X \cdot \partial_+ X - \frac{1}{2}\underbrace{\eta_{++}}_{=0}(\eta^{cd}\partial_c X \cdot \partial_d X) = \dot{X}_L^2 = 0\,, \tag{2.41a}$$
$$T_{--} = \partial_- X \cdot \partial_- X - \frac{1}{2}\underbrace{\eta_{--}}_{=0}(\eta^{cd}\partial_c X \cdot \partial_d X) = \dot{X}_R^2 = 0\,. \tag{2.41b}$$

Oscillator constraints for the closed string

As we saw above, the general solution for the dynamics of the vibrating string can be written in terms of oscillators, so it will be useful to express these constraints in terms of these oscillators. We introduce, for the closed string

$$L_m := \frac{T}{2}\int_0^\pi d\sigma e^{-2im\sigma}T_{--}(0,\sigma) \tag{2.42a}$$
$$= \frac{T}{2}\int_0^\pi d\sigma e^{-2im\sigma}\dot{X}_R^2 \tag{2.42b}$$
$$= \frac{1}{2}\sum_{n=-\infty}^{\infty}\alpha_{m-n}\cdot\alpha_n \tag{2.42c}$$

where for convenience we have defined $\alpha_0^\mu := \frac{1}{2}\ell p^\mu$ and similarly

$$\tilde{L}_m := \frac{1}{2}\sum_{n=-\infty}^{\infty}\tilde{\alpha}_{m-n}\cdot\tilde{\alpha}_n\,, \tag{2.43}$$

where we have again introduced $\tilde{\alpha}_0^\mu := \alpha_0^\mu = \frac{1}{2}\ell p^\mu$. The constraints on the mode expansions are then

$$L_m = \tilde{L}_m = 0 \qquad \forall m \in \mathbb{Z}\,. \tag{2.44}$$

These are known as the (classical) *Virasoro constraints*. The existence of these constraints reflects the fact that our gauge fixing choice $h_{ab} = \eta_{ab}$ does not fully fix all the gauge symmetries of the theory: any reparametrization that sends $\eta_{ab}(\boldsymbol{\sigma}) \to e^{\Theta(\boldsymbol{\sigma})}\eta_{ab}(\boldsymbol{\sigma})$ can be undone with a Weyl transformation, so some gauge invariance remains unfixed. The L_m and $\tilde{L}_m$ are the generators of these transformations. In fact, a little bit of work shows that the remaining unfixed reparametrizations are those generating the conformal group in two dimensions. It is then not a surprise that if one asks what is the algebra generated by the Poisson brackets of the Virasoro constraints we get precisely the Virasoro algebra:

✎ **Exercise 2.6.** *Show that the Virasoro generators L_m, $\tilde{L}_n$ obey the algebra*

$$\{L_m, L_n\}_{P.B.} = -i(m-n)L_{m+n}\,, \tag{2.45a}$$
$$\{\tilde{L}_m, \tilde{L}_n\}_{P.B.} = -i(m-n)\tilde{L}_{m+n}\,, \tag{2.45b}$$
$$\{L_m, \tilde{L}_n\}_{P.B.} = 0\,. \tag{2.45c}$$

Level matching. The constraint $L_0 = \tilde{L}_0$ is particularly important, and receives the name *level matching*. In terms of oscillators, we have

$$\sum_{n=1}^{\infty}\alpha_{-n}\cdot\alpha_n = \sum_{n=1}^{\infty}\tilde{\alpha}_{-n}\cdot\tilde{\alpha}_n\,. \tag{2.46}$$

Note that the spacetime momentum p^μ does not appear here, since $\frac{1}{2}\ell p^\mu = \alpha_0^\mu = \tilde{\alpha}_0^\mu$. This constraint links left and right movers, which are otherwise decoupled in the closed string.

Mass shell condition (closed string). The constraint $L_0 = 0$ implies

$$-\alpha_0 \cdot \alpha_0 = 2 \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \,. \tag{2.47}$$

Note that due to the relativistic mass relation $p \cdot p + M^2 = 0$ and the definition of α_0 we can rewrite this as

$$M^2 = \frac{8}{\ell^2} \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \tag{2.48}$$

or more symmetrically, using the level matching condition

$$\alpha' M^2 = 2 \sum_{n=1}^{\infty} (\alpha_{-n} \cdot \alpha_n + \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n) \tag{2.49}$$

where we have used the constant α' introduced in (2.24). This relation is known as the (closed string) *mass shell condition.* Introducing the mass contribution from left movers and right movers in the obvious way

$$\alpha' M_R^2 \coloneqq 2 \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \tag{2.50a}$$

$$\alpha' M_L^2 \coloneqq 2 \sum_{n=1}^{\infty} \tilde{\alpha}_{-n} \cdot \tilde{\alpha}_n \tag{2.50b}$$

we can alternatively write $M^2 = M_L^2 + M_R^2$, and level matching becomes $M_L^2 = M_R^2$.

Oscillator constraints for the open string

The situation is similar for the open string, with the difference that, as the boundary conditions already link left and right movers, there is no level matching condition. We have a single set of Virasoro generators

$$L_m \coloneqq \frac{1}{2} \sum_{n=-\infty}^{\infty} \alpha_{m-n} \cdot \alpha_n \tag{2.51}$$

where we have defined $\alpha_0^\mu \coloneqq \ell p^\mu$ (note the factor of 2). These generators satisfy the algebra $\{L_m, L_n\}_{\text{P.B.}} = -i(m-n)L_{m+n}$. The mass shell condition for the open string becomes

$$\alpha' M^2 = \sum_{n=1}^{\infty} \alpha_{-n} \cdot \alpha_n \,. \tag{2.52}$$

§3 Quantization of the Polyakov action

§3.1 Canonical quantization

I will now describe how to construct a quantum theory from the classical theory that we have just described. One standard way of doing this is by "canonical quantization". Formally, this quantization procedure amounts to the construction of some linear map Φ from functions in phase space to operators on Hilbert space, such that, ideally

$$[\Phi(f), \Phi(g)] = i\hbar\Phi(\{f, g\}_{\text{P.B.}}) \tag{3.1}$$

for any two functions f, g in phase space. Interestingly, such a map does not exist in general,[1] so we will proceed by defining the action of the map on the elementary operators in the theory, and then study what happens to the commutation relations for more complex operators. We will find, in particular, that the quantized bosonic string can only preserve the Lorentz algebra in D dimensions quantum mechanically if $D = 26$.

We start by introducing quantum operators $\hat{x}^\mu = \Phi(x^\mu)$, $\hat{p}^\mu = \Phi(p^\mu)$, $\hat{\alpha}^\mu_m = \Phi(\alpha^\mu_m)$, $\hat{\tilde{\alpha}}^\mu_m = \Phi(\tilde{\alpha}^\mu_m)$ with commutation relations obtained by canonical quantization of those in (2.38)

$$[\hat{\alpha}^\mu_m, \hat{\alpha}^\nu_n] = [\hat{\tilde{\alpha}}^\mu_m, \hat{\tilde{\alpha}}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\,, \tag{3.2a}$$

$$[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu} \tag{3.2b}$$

with all other commutators vanishing. Here and in what follows I will set $\hbar = 1$.

In order to make $\hat{X}^\mu$ Hermitian we need to impose

$$(\hat{\alpha}^\mu_n)^\dagger = \hat{\alpha}^\mu_{-n} \qquad ; \qquad (\hat{\tilde{\alpha}}^\mu_n)^\dagger = \hat{\tilde{\alpha}}^\mu_{-n}\,. \tag{3.3}$$

This implies that we have an infinite set of raising/lowering operators

$$a^\mu_m := \frac{1}{\sqrt{m}}\hat{\alpha}^\mu_m \qquad m > 0 \tag{3.4a}$$

$$(a^\mu_m)^\dagger := \frac{1}{\sqrt{m}}\hat{\alpha}^\mu_{-m} \qquad m > 0 \tag{3.4b}$$

satisfying the standard algebra

$$[a^\mu_m, (a^\nu_n)^\dagger] = \eta^{\mu\nu}\delta_{m,n} \qquad ; \qquad [a^\mu_m, a^\nu_n] = [(a^\mu_m)^\dagger, (a^\nu_n)^\dagger] = 0 \tag{3.5}$$

and similarly for the left movers.

[1] Under some additional reasonable conditions. This non-existence result is known as Groenewold's theorem.

§3.2 *Ghosts and light-cone gauge fixing*

Given the existence of an oscillator algebra, we can construct the Hilbert space as usual, starting with a vacuum annihilated by all the lowering operators:

$$a_m^\mu \left|0\right\rangle = 0 \tag{3.6}$$

and applying raising operators to it. In addition to the oscillators, we also have the spacetime momentum. We denote the eigenstates of momentum with zero oscillator number by $|0; p^\mu\rangle$.

This Hilbert space that we have just constructed contains *ghosts*: states $(a_m^0)^\dagger \left|0\right\rangle$ with negative norm:

$$|(a_m^0)^\dagger \left|0\right\rangle|^2 = \left\langle 0\right| a_m^0 (a_m^0)^\dagger \left|0\right\rangle = \eta^{00} = -1\,. \tag{3.7}$$

All is not lost, however. In fact, this situation is fairly familiar from quantization of gauge theories, where ghosts are rendered harmless by the gauge invariance. This is also the situation here: as I mentioned above, our gauge fixing $h_{ab} = \eta_{ab}$ is only a partial one, since any reparametrization sending $\eta_{ab}(\boldsymbol{\sigma}) \to e^{\Theta(\boldsymbol{\sigma})}\eta_{ab}(\boldsymbol{\sigma})$ can be composed with a Weyl transformation in order to leave the metric η_{ab} invariant. The infinitesimal form of these transformations was given in (2.9b) and (2.11b). These infinitesimal transformations will cancel (for $h_{ab} = \eta_{ab}$) if

$$\partial^a \xi^b + \partial^b \xi^a = \Theta \eta^{ab}\,. \tag{3.8}$$

In the $+-$ coordinate system these equations imply $\partial_+\xi^- = \partial_-\xi^+ = 0$ and $\Theta = \partial_+\xi^+ + \partial_-\xi^-$, with general solution $\xi^\pm = \xi^\pm(\sigma^\pm)$.[2] Since $X^\mu \to X^\mu + \xi^a\partial_a X^\mu$ under reparametrizations, we can use this remaining gauge symmetry to choose a convenient parametrization that removes all classical oscillator modes in one of the spacetime directions. When we canonically quantize the resulting theory we will find a Hilbert space without ghosts.

The most convenient choice is perhaps surprising. Let me go back to the classical theory for a moment in order to do the gauge fixing, and introduce so-called *light-cone coordinates* in the target spacetime:

$$X^\pm = \frac{1}{\sqrt{2}}(X^0 \pm X^1)\,. \tag{3.9}$$

We leave the remaining $D-2$ coordinates invariant, and will use "i" to index these last $D-2$ coordinates. In these coordinates indices are raised and lowered by

$$V^+ = -V_- \qquad ; \qquad V^- = -V_+ \qquad ; \qquad V^i = V_i\,. \tag{3.10}$$

and the dot product is given by

$$U \cdot V = U^i V_i - U^+ V^- - U^- V^+\,. \tag{3.11}$$

[2]Note that if you complexify the worldsheet coordinates, the statement is that the remaining symmetries are $z \to z + f(z)$, which explains the appearance of the Virasoro generators of the conformal group as constraints in our discussion of the classical theory.

We will then use our remaining gauge invariance to set (for the closed string, the open string case is entirely analogous so we will not treat it separately)

$$X_R^+ = \frac{1}{2}x^+ + \frac{1}{2}\ell^2 p^+ \sigma^- + \frac{i}{2}\ell \sum_{n\neq 0} \frac{1}{n}\alpha_n^+ e^{-2in\sigma^-} \tag{3.12}$$

to

$$X_R^+ = \frac{1}{2}\ell^2 p^+ \sigma^- \tag{3.13}$$

and similarly for the left movers, so we end up with

$$X^+ = \ell^2 p^+ \tau\,. \tag{3.14}$$

That is, we have set $x^+ = \alpha_m^+ = 0$ by a change of coordinates in the worldsheet. Note that we do not set $X^+ = 0$ fully: to eliminate the term proportional to τ in X^+ we would need to choose $\xi^\pm(\sigma^\pm) = -\sigma^\pm$, which would give $\sigma^\pm \to \sigma^\pm - \sigma^\pm = 0$, so this is not a reparametrization that we will want to consider. (Note also that the formulas below become singular when $p^+ \to \infty$.)

In the light cone gauge we have $\dot{X}^+ = \ell^2 p^+$ and $X'^+ = 0$ so the $(\dot{X} \pm X')^2 = 0$ constraint can be rewritten as

$$\sum_i (\dot{X}^i \pm X'^i)^2 = 2\ell^2 p^+ (\dot{X}^- \pm X'^-) \tag{3.15}$$

where here and in what follows the sum in i is over $2, \ldots, D-1$. With the exception of the constant term x^-, these two equations allow us to completely solve for X^- in terms of the X^i, so we no longer have independent oscillator modes in this direction either.

✎ **Exercise 3.1.** *Show that for the open string*

$$\alpha_m^- = \frac{1}{2\ell p^+} \sum_{n=-\infty}^{\infty} \sum_i \alpha_{m-n}^i \alpha_n^i\,. \tag{3.16}$$

Similarly, if we express the Virasoro generators in the spacetime lightcone coordinates we have

$$L_m = \frac{1}{2} \sum_{n=-\infty}^{\infty} \left(\sum_i \alpha_{m-n}^i \alpha_n^i - \alpha_{m-n}^+ \alpha_n^- - \alpha_{m-n}^- \alpha_n^+ \right) \tag{3.17}$$

and similarly for the left-movers. The $m = 0$ case is particularly important, as it leads to the level matching and mass shell conditions, which in light cone gauge read (for the closed string) $M_L^2 = M_R^2$ and $M^2 = M_L^2 + M_R^2$, with

$$\alpha' M_R^2 = 2 \sum_{n=1}^{\infty} \sum_i \alpha_{-n}^i \alpha_n^i \tag{3.18a}$$

$$\alpha' M_L^2 = 2 \sum_{n=1}^{\infty} \sum_i \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i\,. \tag{3.18b}$$

Note that for the sum is over transverse oscillators only. For the open string we have similarly

$$\alpha' M^2 = \sum_{n=1}^{\infty} \sum_i \alpha^i_{-n} \alpha^i_n \,. \tag{3.19}$$

§3.3 *Quantization in the light cone gauge and ordering ambiguities*

We learn in this way that, effectively, in the light-cone gauge the independent oscillator degrees of freedom in the classical solution are only the transverse ones, namely the α^i_m. In order to canonically quantize this theory we would need to compute the Poisson brackets after having fixed the light cone gauge. I will leave this to the enterprising readers, and I will just quote the unsurprising result for the non-vanishing commutators in the quantum theory:

$$[\hat\alpha^i_m, \hat\alpha^j_n] = [\hat{\tilde\alpha}^i_m, \hat{\tilde\alpha}^j_n] = m\delta_{m+n}\delta^{i,j}\,, \tag{3.20a}$$

$$[\hat x^i, \hat p^j] = i\delta^{i,j}\,, \tag{3.20b}$$

$$[\hat x^-, \hat p^+] = -i\,. \tag{3.20c}$$

with all other commutators vanishing. I have not included commutators for the $\hat\alpha^-_m$ here, since in the light cone gauge the $\hat\alpha^-$ are constructed in terms of transverse oscillators, with the explicit expression given in exercise (3.1), so they are to be understood as composite operators.

These commutators determine the commutator relations of any operator built out these oscillators, but there is an important ambiguity that we need to deal with: in the classical theory the mass shell and level matching conditions involve an ordering ambiguity, since $\hat\alpha^i_{-n}\hat\alpha^i_n \neq \hat\alpha^i_n\hat\alpha^i_{-n}$, so in principle the expression in the spacetime mass in the quantum theory could include a c-number contribution

$$\alpha'\hat M^2_R = \left(2\sum_{n=1}^{\infty}\sum_i \hat\alpha^i_{-n}\hat\alpha^i_n\right) - 2a_R \tag{3.21a}$$

$$\alpha'\hat M^2_L = \left(2\sum_{n=1}^{\infty}\sum_i \hat{\tilde\alpha}^i_{-n}\hat{\tilde\alpha}^i_n\right) - 2a_L \tag{3.21b}$$

for some unknown constants a_R and a_L. There is, unfortunately, no simple and rigorous argument that fixes these constants, so I will start by giving a simple but heuristic argument, and then outline one rigorous, but technically more involved, computation that leads to the same results.[3]

[3]Other approaches leading to the same results exist, the most satisfying probably being BRST quantization of the string worldsheet theory.

A heuristic computation using that $1 + 2 + 3 + \ldots = -\frac{1}{12}$

The basic indeterminacy comes from the fact that we need to determine the right action of the quantization map Φ on the quadratic operators $\alpha^i_{-n}\alpha^i_n$. Since $[\Phi(\alpha^i_{-n}), \Phi(\alpha^i_n)] \neq 0$, it is not obvious which is the right choice, with a reasonable definition of "right" being that Φ should preserve as many symmetries of the classical theory as possible, or equivalently (3.1) is satisfied for as many functions in phase space as possible. In the next section we will determine $a_{L,R}$ by imposing that Lorentz invariance on the target spacetime is preserved quantum mechanically, or in other words that the Poisson brackets between the conserved charges under Lorentz transformations are preserved quantum mechanically. But before going into that, let us try a simple guess motivated by the existence of a simple choice for Φ that is known to preserve a large amount of the Poisson bracket structure, known as the *Weyl quantization rule.*[4]

Say that you have a pair of canonically conjugate variables q, p, and a function in phase space $f(q,p)$. Then the Weyl quantization operator Φ_W acts as

$$\Phi_W[f] = \frac{1}{(2\pi)^2}\int f(q,p)e^{i(a(p-\hat{p})+b(q-\hat{q}))}dpdqdadb \tag{3.22}$$

where $\hat{q}$ and $\hat{p}$ are the fundamental position and momentum operators acting on the Hilbert space, with commutator $[\hat{q},\hat{p}] = i$. The map Φ_W has a number of remarkable properties, but we will only need to use that it is linear, and that

$$\Phi_W[q^n] = \hat{q}^n \qquad ; \qquad \Phi_W[p^n] = \hat{p}^n\,, \tag{3.23}$$

as it can be shown easily. We can construct such canonically conjugate variables out of our oscillators α^i_n via (we assume $n > 0$ here)

$$q^i_n = \frac{1}{\sqrt{2n}}(\alpha^i_n + \alpha^i_{-n}) \tag{3.24a}$$

$$p^i_n = \frac{1}{i\sqrt{2n}}(\alpha^i_n - \alpha^i_{-n})\,. \tag{3.24b}$$

These variables have $\{q^i_n, p^j_m\}_{\text{P.B.}} = \delta^{i,j}\delta_{m,n}$. In terms of these canonical variables we have

$$\alpha^i_n\alpha^i_{-n} = \frac{n}{2}\left((q^i_n)^2 + (p^i_n)^2\right)\,, \tag{3.25}$$

so Weyl's quantization map immediately implies

$$\Phi_W(\alpha^i_n\alpha^i_{-n}) = \frac{n}{2}\left((\hat{q}^i_n)^2 + (\hat{p}^i_n)^2\right) = \frac{1}{2}(\hat{\alpha}^i_n\hat{\alpha}^i_{-n} + \hat{\alpha}^i_{-n}\hat{\alpha}^i_n)\,. \tag{3.26}$$

[4] In particular, the Weyl quantization rule is known to preserve the Poisson brackets of any quadratic function f in phase space with any arbitrary order function g, so it is natural to expect that the quantum constraint leading to the mass shell condition is $\Phi_W(L_0) = 0$.

For the mass itself we thus obtain

$$\begin{aligned}\Phi_W(\alpha' M_R^2) &= \Phi_W\left(2\sum_i \sum_{n=1}^{\infty} \alpha_{-n}^i \alpha_n^i\right) \\ &= 2\sum_i \frac{1}{2}\sum_{n\neq 0} \hat{\alpha}_{-n}^i \hat{\alpha}_n^i \\ &= 2\sum_i \frac{1}{2}\sum_{n=1}^{\infty} (\hat{\alpha}_n^i \hat{\alpha}_{-n}^i + \hat{\alpha}_{-n}^i \hat{\alpha}_n^i) \\ &= 2\left(\sum_{n=1}^{\infty}\sum_i \hat{\alpha}_{-n}^i \hat{\alpha}_n^i + \frac{D-2}{2}\sum_{n=1}^{\infty} n\right)\end{aligned} \tag{3.27}$$

where in the last line we have used the commutation relation $[\hat{\alpha}_n^i, \hat{\alpha}_{-n}^i] = n$. Comparison with (3.21), using a completely analogous argument for the left movers, shows that

$$a_L = a_R = -\frac{(D-2)}{2}\sum_{n=1}^{\infty} n \tag{3.28}$$

The sum on the right hand side is ill-defined, so here comes the step that makes this argument somewhat heuristic: let me define the right hand side in terms of the Riemann ζ function

$$\zeta(s) := \sum_{n=1}^{\infty} \frac{1}{n^s} \tag{3.29}$$

where the right hand side is well defined for $\Re(s) > 1$, and has a unique analytic continuation to the whole complex plane. This choice is very natural, and in fact one can motivate it better physically by introducing a cutoff in the worldsheet theory, and adding suitable counterterms. At any rate, once we accept this, we have

$$a_L = a_R = -\frac{(D-2)}{2}\zeta(-1) = \frac{D-2}{24}\,. \tag{3.30}$$

If we introduce the number operators

$$\hat{N}_R := \sum_{n=1}^{\infty}\sum_i \hat{\alpha}_{-n}^i \hat{\alpha}_n^i \qquad ; \qquad \hat{N}_L := \sum_{n=1}^{\infty}\sum_i \hat{\tilde{\alpha}}_{-n}^i \hat{\tilde{\alpha}}_n^i \tag{3.31}$$

we have for the closed string

$$\alpha' \hat{M}^2 = 2(\hat{N}_R + \hat{N}_L) - \frac{D-2}{6} \tag{3.32}$$

subject to the level matching condition $\hat{N}_L - \hat{N}_R = 0$.

An analogous argument, taking into account the factors of two, tells us that for the open string we have

$$\alpha' \hat{M}^2 = \hat{N} - a \tag{3.33}$$

with

$$\hat{N} := \sum_{n=1}^{\infty} \sum_{i} \hat{\alpha}^i_{-n} \hat{\alpha}^i_n \qquad \text{and} \qquad a = \frac{D-2}{24} \, . \tag{3.34}$$

§3.4 *The light spectrum of the bosonic string*

We have now all the tools that we need to construct the spacetime spectrum of the bosonic string. Let me start by studying the open string. We have a vacuum $|0\rangle$. We have

$$\alpha' \hat{M}^2 |0\rangle = (\hat{N} - \frac{D-2}{24}) |0\rangle = -\frac{D-2}{24} |0\rangle \tag{3.35}$$

so this state is *tachyonic* for $D > 2$. We reach the same conclusion for the vacuum for the closed string, with mass

$$\alpha' \hat{M}^2 |0\rangle = -\frac{D-2}{6} |0\rangle \, . \tag{3.36}$$

A tachyon in a field theory is not an inconsistency: it just means that we are doing perturbation expansion around a maximum of the potential, instead of a minimum. The Higgs field in the standard model is an example: we certainly do not want to discard the standard model because it has a tachyon! Rather, what we need to do is to follow the fate of the tachyon as it rolls down the potential until it settles in the minimum, if one exists. While we understand the fate of tachyonic modes in string theory in some cases,[5] the case of the tachyon in the closed string sector of the bosonic string is still open, and we do not know what happens to it for large tachyonic vacuum expectation values. There are various possibilities that we could imagine: the bosonic string might decay to one of the supersymmetric strings, the string might confine so the theory becomes trivial, a theory without a weakly coupled gravitational interpretation might arise on spacetime, or perhaps the bosonic string simply does not make sense non-perturbatively for some reason that is not apparent in the perturbative description. It is an important and interest question to understand what happens to the bosonic string under tachyon condensation, but we do not yet have the technology to understand what is going on. One of our motivations for introducing the superstring will be in fact to get rid of this mysterious tachyon in the spectrum.

Let us move to the first excited states. In the case of the open string these are the states $\hat{\alpha}^i_{-1} |0\rangle$, for $i \in \{2, \ldots, D-1\}$, with mass

$$\alpha' \hat{M}^2 \hat{\alpha}^i_{-1} |0\rangle = \left(1 - \frac{D-2}{24}\right) \hat{\alpha}^i_{-1} |0\rangle \, . \tag{3.37}$$

[5] In particular, we have a reasonably good handle on the problem in the case of open strings, see Sen's review arXiv:hep-th/0410103 for a good overview.

These $D-2$ states transform as vectors of the $SO(D-2)$ group rotating the transverse directions. We would like the full theory to be invariant not only under this group, but under the full $SO(D-1,1)$ Lorentz group of the target spacetime. A massive vector particle in a Lorentz-invariant theory in D dimensions has $D-1$ physical polarizations, while a massless particle has $D-2$ physical polarizations. More generally, massive particles transform in representations of the little group $SO(D-1)$, while massless particles transform in representation of $SO(D-2)$.[6] It must therefore be the case, if we want Lorentz invariance in the target space to hold, that the set of states that we have just found are massless, which is only the case if $D = 26$. This is known as the *critical dimension* for the bosonic string.

So we have found that the mass formula for open string states in the critical dimension $D = 26$ is

$$\boxed{\alpha' \hat{M}^2 = \hat{N} - 1} \tag{3.38}$$

while for the closed string states we have

$$\boxed{\alpha' \hat{M}^2 = 2(\hat{N}_R + \hat{N}_L) - 4} \tag{3.39}$$

The first excited states in the closed string sector can be analysed similarly. Recalling that we need to impose the level matching condition $\hat{N}_L = \hat{N}_R$, the massless states are of the form $|ij\rangle \coloneqq \alpha^i_{-1}\tilde{\alpha}^j_{-1}|0\rangle$. These states transform in the $\mathbf{24}\otimes\mathbf{24}$ representation of $SO(24)$, which decomposes as a symmetric traceless tensor (the $\mathbf{299}$ irrep of $SO(24)$) G_{ij}, an antisymmetric two-form B_{ij} (in the $\mathbf{276}$ of $SO(24)$) and a singlet Φ. An alternative way of describing the G_{ij} representation is as a spin-two excitation, which is the same representation as the graviton in a theory in gravity. And indeed, this field couples to the other fields in the spectrum as a metric (more on this below). So we identify this excitation with the graviton in the target spacetime.

This is something very profound: the bosonic string *requires* gravity in the target spacetime to exist! Finding a consistent quantum theory of gravity is a notoriously difficult problem, and it was a very fortunate discovery that string theory, a theory originally developed for the purpose of understanding nuclear interactions, turned out to be one such theory.[7]

The other two fields at the massless level are also interesting. B_{ij} the theory of a massless 2-form field, sometimes known as the "Kalb-Ramond" field, or often simply the "B" field. A Lorentz invariant description of such fields requires the introduction of a gauge equivalence $B \to B + d\lambda$, where λ is a 1-form. Finally, Φ is known as the "dilaton".

[6] Section 2.5 in Weinberg's book has a systematic discussion of this point, if you have not seen it before in your studies.

[7] The connection between string theory and nuclear interactions was revived with the discovery of the AdS/CFT correspondence, which connects strongly coupled field theory phenomena with gravitational ones.

✎ **Exercise 3.2.** *Construct the states at the next mass level in the open and closed string sectors. As these are massive, they should assemble into representations of the massive little group $SO(25)$. Show that this is indeed the case.*

§3.5 A sketch of a more rigorous derivation of the critical dimension

There is a different way of reaching the same conclusion which avoids the need of worrying about the infinite summation step, at the expense of involving a rather more significant amount of algebra. I will only provide the outline of the argument, but those of you who are curious about the details are encouraged to try filling them in. You can find a more detailed description of the computation in the book by Green, Schwarz and Witten, and a fully worked out derivation in the lecture notes on string theory by Gleb Arutyunov.

We have seen above that in the light cone gauge there is an issue with preserving Lorentz invariance in the spectrum unless $D = 26$ and $a_L = a_R = 1$. It is reasonable to expect that the same issue will manifest itself already at the level of the algebra of Lorentz generators in spacetime. Recall that the classical Lorentz algebra has commutators

$$[J^{\mu\nu}, J^{\sigma\rho}] = i(\eta^{\nu\rho}J^{\mu\sigma} - \eta^{\mu\rho}J^{\nu\sigma} - \eta^{\nu\sigma}J^{\mu\rho} + \eta^{\mu\rho}J^{\nu\sigma})\,. \tag{3.40}$$

These commutation relations should be reproduced by the Poisson brackets of the classical conserved charges constructed in exercise 2.5, and a relatively short computation shows that this is indeed the case.

As pointed out in footnote 4, Weyl quantization preserves the bracket structure for any two phase space functions which are quadratic on the phase space variables, so naively it seems like the quantum theory should preserve the Lorentz algebra too, since the expressions in exercise 2.5 are indeed quadratic. But this is a bit too quick: while the $J^{\mu\nu}$ found in that exercise are indeed quadratic on the α^μ oscillators, in the light cone gauge

$$\alpha_m^- = \frac{1}{2\ell^2 p^+}\sum_{n=-\infty}^{\infty}\sum_i \alpha^i_{m-n}\alpha^i_n \tag{3.41}$$

as you derived in exercise (3.1). So the J^{i-} conserved charges are *cubic* on the physical light cone oscillators.

This implies in particular that the classical commutation relations involving $J^{\mu-}$ are not necessarily preserved by the Weyl quantization map. And indeed, classically we have $[J^{i-}, J^{j-}] = 0$, as you can check easily from the expression above, but a (long) computation shows that quantum mechanically

$$[\hat{J}^{i-}, \hat{J}^{j-}] = -\frac{1}{(\ell^2 p^+)^2}\sum_{m=1}^{\infty}\Delta_m(\hat{\alpha}^i_{-m}\hat{\alpha}^j_m - \hat{\alpha}^j_{-m}\hat{\alpha}^i_m) \tag{3.42}$$

with

$$\Delta_m = m\left(\frac{26-D}{12}\right) + \frac{1}{m}\left(\frac{D-26}{12} + 2(1-a)\right) \tag{3.43}$$

where a is the same ordering constant that appeared in (3.33). We have that $\Delta_m = 0$ only for $D = 26$ and $a = 1$, in agreement with the results from the ζ function argument above.

§3.6 *Interactions in string theory*

Before we move on to the superstring, I would like to discuss a little bit more in detail how the massless closed string fields couple to the worldsheet. As I mentioned above, the field $G_{\mu\nu}$ can be interpreted as the dynamical excitations of the flat background metric, which so far we have chosen to be $\eta_{\mu\nu}$. (In this section we will momentarily undo all of the gauge fixings that we have done above to bring the Polyakov action to a more manageable form.) This interpretation allows us to make a natural guess for how to describe the propagation of strings on curved manifolds, with background metric $G_{\mu\nu}(X)$. It is simply

$$S_P = -\frac{T}{2}\int_\Sigma d^2\boldsymbol{\sigma}\sqrt{h}h^{ab}\partial_a X^\mu \partial_b X^\nu G_{\mu\nu}(X)\,. \tag{3.44}$$

It is easy to see that this action is again equivalent on shell to the induced area of Σ. But note that from the point of view of the worldsheet this is a rather drastic modification: since $G_{\mu\nu}(X)$ is now a non-trivial function of the embedding coordinates the theory on the worldsheet is no longer a free theory, since the coefficient of the kinetic term for the fields is a non-linear function of the fields, and this introduces interactions. Whenever the curvature is small we can try do perturbation theory to deal with the leading modifications induced by these interactions, but for arbitrary curved metrics we typically we cannot solve the theory on the string anymore. I am definitely not saying that we cannot say anything about string theory on curved manifolds! But the worldsheet perspective is typically less useful here.

The coupling to the two-form $B_{\mu\nu}$ is of the form

$$S_B = -\frac{T}{2}\int_\Sigma d^2\boldsymbol{\sigma}\epsilon^{ab}\partial_a X^\mu \partial_b X^\nu B_{\mu\nu}(X) \tag{3.45}$$

where ϵ is the antisymmetric symbol in two indices, with $\epsilon^{01} = -\epsilon^{10} = 1$ and $\epsilon^{00} = \epsilon^{11} = 0$. In terms of the embedding into spacetime, this term is just the integral of B over the worlsheet Σ

$$S_B = -\frac{T}{2}\int_\Sigma B \tag{3.46}$$

where now Σ should be understood as a submanifold in spacetime, and $B_{\mu\nu}$ is a form on spacetime. This kind of term is the two-dimensional generalization of the coupling to a background field for a charged particle, which is of the form

$$S_e = q\int_\gamma A \tag{3.47}$$

where q is the charge of the particle, γ its worldline, and A the background 1-form $U(1)$ connection for the electromagnetic field. Because of this, we generally say that the fundamental string is electrically charged under B.

The final massless field of the bosonic string, the dilaton Φ, is initially more perplexing. It couples to the worldsheet in a conformally invariant way by

$$S_\Phi = \frac{1}{4\pi}\int_\Sigma d^2\boldsymbol{\sigma}\sqrt{h}\Phi(X)R_h\,. \tag{3.48}$$

Here R_h is the two dimensional Ricci scalar. The existence of this coupling has rather important consequences. First, note that by the Gauss-Bonnet theorem, if Σ is a closed surface,

$$\frac{1}{4\pi}\int_\Sigma d^2\boldsymbol{\sigma}\sqrt{h}R_h = 2-2g(\Sigma) \tag{3.49}$$

where $g(\Sigma)$ is the *genus* of Σ, a topological invariant. This implies that for constant Φ we have

$$S_\Phi = \Phi(2-2g(\Sigma)) \tag{3.50}$$

assuming that Σ is a closed surface.

In order to understand the implications of this, let us consider the loop expansion of the interaction of three particles in the target spacetime, as in figure 3.

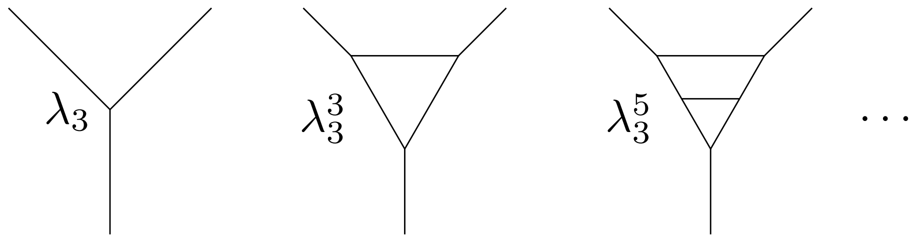
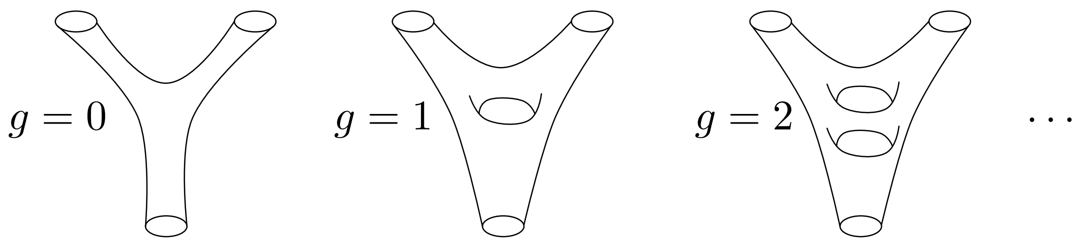


Figure 3: Some of the leading diagrams in a loop expansion of a cubic interaction in spacetime.

We do not need to be very precise about the nature of the interactions or particle content here to see the main point, so let us just assume that all particles in the theory are of the same type, and there is a single cubic interaction to worry about, with coupling constant λ_3. We have seen during the lectures that in string theory spacetime particles can be understood as particular excitations of a single string, so in string theory the spacetime diagrams in figure 3 should be rather understood as particular worldsheet topologies (where each propagating particle is "thickened" into a string with worldsheet locally a cylinder). The resulting string diagram representing the spacetime interaction will look something like the following picture:



(3.51)

There are various things that are noteworthy about this diagram. First, note that there is no point in the string worldsheet that is not smooth. In other words: there is no local notion of interaction in string theory, it is only the global topology of the worldsheet that tells us something about interactions in the field theory limit. A local observer in the string worldsheet just sees free propagation locally. This is a very important distinction with respect to the case of the propagating particle: while in the case of the propagating particle any possible interaction vertex required us to provide one extra independent piece of data (the coupling constant, as in figure 1), in the case of the string all the spacetime interactions are in principle fixed from the behaviour of the freely propagating string. Or in other words, there are no adjustable coupling constants in string theory.

Note also that adding loops in the spacetime diagram corresponds to increasing the genus of the surface on which the string is propagating: tree level interactions correspond to strings on Riemann surfaces of genus 0, one loop interactions correspond to strings with worldsheets of genus one, and in general l-loop processes in the particle picture correspond to string diagrams of genus $g = l$.

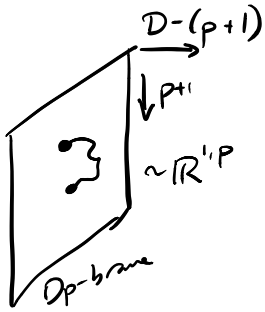
Here is where the dilaton Φ enters the story: note that the particle interaction is weighted by λ_3^{1+2l}, with l the number of loops in the diagram, while (for constant background dilaton) the (euclidean) string action is weighted by $e^{-\Phi(2-2g)}$. And we have just argued that $g = l$. This implies that by changing the vacuum expectation value of Φ we can change the effective spacetime coupling λ_3. More precisely, $\langle\Phi\rangle \to \langle\Phi\rangle + c$ will act in the field theory limit as $\lambda_3 \to e^c\lambda_3$. So there are no adjustable couplings in string theory, but this does not mean that all the couplings in the effective theory are determined at the outset. Rather, in string theory every coupling "constant" of the effective field theory is secretly the vacuum expectation value of some dynamical background field.

§4 D-branes

So far we have imposed Neumann boundary conditions on both ends of the open string in order to preserve Poincaré invariance in the target spacetime. While this might have been a natural assumption in the early days of string theory, the perspective has changed during the last couple of decades, and D-branes (the result of considering Dirichlet boundary conditions instead) have becomes more and more important in our understanding of string theory.

Definition 4.1. We say that we have a *Dp-brane* in a D-dimensional theory if we are imposing Dirichlet boundary conditions $\delta X = 0$ on $D - p - 1$ directions of spacetime, and Neumann boundary conditions $X' = 0$ on the remaining $p + 1$ directions.

That is, the endpoints of the open string are restricted to live on a $p + 1$ submanifold of spacetime, and we say that we have a Dp-brane wrapping the $p + 1$ submanifold where open strings can end. (Whenever we do not want to emphasize the number of directions where we impose Dirichlet boundary conditions we speak of "D-branes" instead, without specifying the precise value of p.)



In general, to specify the position of a Dp-brane inside some ambient space we need to give $D - (p + 1)$ equations. We will consider the simplest case in which the D-brane is wrapping a $\mathbb{R}^{1,p}$ submanifold of flat spacetime $\mathbb{R}^{1,D-1}$ (this is certainly not the most general case, and deep insights into geometry and quantum field theory follow from considering branes wrapping more involved subspaces). So we split the spacetime coordinates into directions along the brane, and directions orthogonal to it, as follows:

$$\underbrace{X^+, X^-, X^{i=2}, \ldots, X^{i=p}}_{p+1 \text{ directions along the brane}}, \qquad \underbrace{X^{a=p+1}, \ldots, X^{a=D-1}}_{D-(p+1) \text{ transverse directions}} \tag{4.1}$$

where we have introduced an index "a" running over the Dirichlet directions and an index "i" running over the Neumann directions transverse to the lightcone coordinates. The equations defining the geometric subspace where the Dp-brane lives are therefore:

$$X^a = x_1^a \tag{4.2}$$

with $x_1^a \in \mathbb{R}$ some constants encoding the position of the D-brane. We will also assume $p \geq 1$, so we can keep using lightcone methods. One can also study the $p = 0$ and $p = -1$ cases using more sophisticated methods.

In summary, for the case of a single Dp-brane (we will study the generalisation to multiple Dp-branes below) we have

$$\begin{aligned} (X')^{\pm}(\tau, 0) = (X')^{\pm}(\tau, \pi) = 0 \\ (X')^{i}(\tau, 0) = (X')^{i}(\tau, \pi) = 0 \qquad \text{for } i \in \{2, \dots, p\} \end{aligned} \tag{4.3}$$

for the Neumann-Neumann (henceforth "NN" for brevity) directions,[8] and

$$X^a(\tau, 0) = X^a(\tau, \pi) = x_1^a \qquad \text{for } a \in \{p+1, \dots, D-1\} \tag{4.4}$$

for the Dirichlet-Dirichlet (DD) directions. We will focus on the description of D-branes in the bosonic string, the discussion generalises fairly straightforwardly to the superstring.

§4.1 Classical aspects

The mode expansion in the NN directions works just as before, so let us concentrate on the DD mode expansion. Away from the boundaries we still have the free wave equation, with standard D'Alembert solution

$$X^a(\tau, \sigma) = X_R^a(\sigma^-) + X_L^a(\sigma^+)\,. \tag{4.5}$$

At $\sigma = 0$ the boundary condition $X^a(\tau, 0) = x_1^a$ implies

$$X_R^a(\tau) + X_L^a(\tau) = x_1^a \tag{4.6}$$

so we can solve $X_L^a(\tau) = x_1^a - X_R^a(\tau)$. Similarly at $\sigma = \pi$ we have $X^a(\tau, \pi) = x_1^a$, which implies

$$X_R^a(\tau - \pi) + X_L^a(\tau + \pi) = X_R^a(\tau - \pi) + (x_1^a - X_R^a(\tau + \pi)) = x_1^a\,, \tag{4.7}$$

so we learn that $X_R^a(\tau - \pi) = X_R^a(\tau + \pi)$, so X_R^a is a periodic function with period 2π, and thus admits an expansion in terms of sines and cosines:

$$X_R^a(\sigma^-) = x_R^a + \sum_{i=1}^{\infty} (s_n^a \sin(n\sigma^-) + c_n^a \cos(n\sigma^-)) \tag{4.8}$$

which implies, due to (4.6)

$$X_L^a(\sigma^+) = x_L^a - \sum_{i=1}^{\infty} (s_n^a \sin(n\sigma^+) + c_n^a \cos(n\sigma^+)) \tag{4.9}$$

[8] We repeat "Neumann" here to indicate that the boundary conditions are Neumann on both ends of the string. It is possible to consider configurations in which one end of the string has Neumann boundary conditions and the other has Dirichlet boundary conditions, we refer to these as "Dirichlet-Neumann" or "Neumann-Dirichlet" boundary conditions, or often simply "DN" and "ND".

with x_L^a and x_R^a constants satisfying $x_L^a + x_R^a = x_1^a$. From these solutions for the left and right movers, and some easy redefinitions, we find

$$X^a(\tau,\sigma) = x_1^a + \ell \sum_{n\neq 0} \frac{1}{n}\alpha_n^a \sin(n\sigma) e^{-in\tau}\,. \tag{4.10}$$

We see that there are some noteworthy differences with respect to the NN result (2.26). First, and most obviously, the choice of boundary conditions turns the cosine into a sine. More interestingly, the center of mass momentum p^a is absent. This is reasonable, since we are restricting the endpoints of the string to $X^a = x_0^a$, so there is no zero mode of constant momentum along that direction. Finally, there is an overall factor of i missing from the sum: this is a choice of convention so that the reality condition is $(\alpha_n^a)^* = \alpha_{-n}^a$ as usual.

✎ **Exercise 4.1.** *Show that the Poisson brackets in the DD sector are*

$$\{\alpha_m^a, \alpha_n^b\} = -im\delta_{m+n}\delta^{a,b}\,. \tag{4.11}$$

Note that there is no Poisson bracket involving x_1^a: this is a constant and not a phase space variable, which is compatible with the fact that the corresponding momentum p^a is absent.

✎ **Exercise 4.2.** *Write down the mode expansion for the DN sector. You should obtain*

$$X^a(\tau,\sigma) = x_1^a + \ell \sum_{r\in\mathbb{Z}+\frac{1}{2}} \frac{1}{r}\alpha_r^a \sin(r\sigma) e^{-ir\tau} \tag{4.12}$$

with $x_1^a \in \mathbb{R}$ a constant. What is the reality condition on the α_r? Show that the Poisson brackets are

$$\{\alpha_s^a, \alpha_r^b\} = -is\delta_{r+s}\delta^{a,b}\,. \tag{4.13}$$

§4.2 The quantum theory

Quantization of the DD sector is straightforward, and it works very similarly to the NN sector. The main difference is that there is no p^a momentum, so while the mass formula is still

$$\alpha'\hat{M}^2 = \hat{N} - 1 \tag{4.14}$$

with

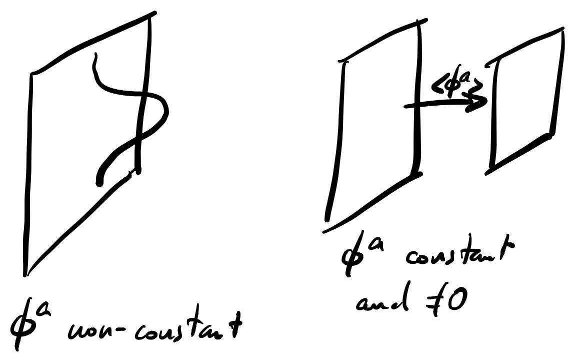
$$\hat{N} := \sum_{n=1}^{\infty}\left(\left(\sum_{i=2}^{p}\hat{\alpha}_{-n}^i\hat{\alpha}_n^i\right) + \left(\sum_{a=p+1}^{D-1}\hat{\alpha}_{-n}^a\hat{\alpha}_n^a\right)\right) \tag{4.15}$$

the mass $\hat{M}$ should be interpreted as the mass of a particle moving in $p+1$ dimensions only:

$$M^2 + \left(\sum_{i=2}^{p+1} p^i p^i\right) - 2p^+p^- = 0\,. \tag{4.16}$$

Let us compute the spectrum. In the ground state we find a tachyon with $\alpha'\hat{M}^2 = -1$. This tachyon is removed in the superstring, so let us not discuss it further.[9] The first excited states $\hat{\alpha}^i_{-1}|0\rangle$ are massless, and transform as vectors of the $SO(p-1)$ massless little group on the brane, so we identify them as gauge bosons of a $U(1)$ gauge symmetry.

We also find $D-(p+1)$ massless scalars ϕ^a coming from $\hat{\alpha}^a_{-1}|0\rangle$. These do not transform under the little group on the brane, but they transform as a vector of $SO(D-(p+1))$, which from the point of view of the $(p+1)$-dimensional theory living on the brane is a global symmetry. In fact, these ϕ^a modes have a very important interpretation: they encode deformations of the brane in the transverse dimensions. In fact, the effect of giving a vacuum expectation value $\langle\phi^a\rangle = v^a \neq 0$ in the $(p+1)$-dimensional theory of the brane is to move the brane from x_1^a to $x_1^a + v^a$.



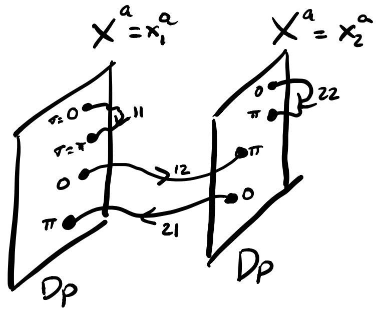
So, once again we find (recall our discussion in §3.6) that x_1^a, which initially seemed like an external parameter to the theory, turns out to be the expectation value of a dynamical field.

[9]Although the proofs are fairly technical, the fate of this open string tachyon in the bosonic string is much better understood: the brane dissolves into closed string radiation. See for instance hep-th/0410103 and arXiv:1912.00521 for reviews.

§4.3 Multiple parallel D-branes

We can extend the previous discussion to the case of multiple D-branes. What this means is that we will consider open strings whose two endpoints are not necessarily at the same position. We rather have a set of N positions x_i^a with $i = 1, \ldots, N$ such that each of the two endpoints of the open string can be on any of these positions. (This set of choices of endpoint is sometimes known as the "Chan-Paton factors" for the open string.) We speak of the ij sector to indicate the choice of boundary condition $X^a(\tau, 0) = x_i^a$, $X^a(\tau, \pi) = x_j^a$, and we say that we have N D-branes at the positions x_i^a.

We will assume for simplicity that the multiple branes have the same dimension, and that they are parallel. There is no logical requirement for this to be so, and dropping these requirements leads to very interesting physics. I encourage the enterprising student to try to work out what happens in these cases. We will also assume for the moment that $x_m^a \neq x_n^a$ for $m \neq n$, although we will see later that something rather interesting happens if we drop the assumption.



Let us start with the case of 2 Dp-branes, separated in the X^a directions. The NN sectors work as before, but in the DD sectors we have four possibilities, depending on which position we choose for the open string endpoints:

11. In the "11 sector" for the open string we take $X^a(\tau, 0) = X^a(\tau, \pi) = x_1^a$. The discussion proceeds exactly as in the previous section, so we end up with a $U(1)$ gauge boson, and $D - (p + 1)$ scalars.

22. In the "22 sector" we take $X^a(\tau, 0) = X^a(\tau, \pi) = x_2^a$, so we obtain another (independent) $U(1)$ gauge boson and $D - (p + 1)$ scalars.

The 12 and 21 sectors are new. They are interpreted as strings stretched between branes at x_1^a and x_2^a. Because the open string carries an orientation, these are two independent (but closely related) sectors. The analysis is familiar by now, so I will leave it as an exercise:

✎ **Exercise 4.3.** *Show that the mode expansion in the ij sector is*

$$X^a(\tau, \sigma) = x_i^a + \frac{\sigma}{\pi}(x_j^a - x_i^a) + \ell \sum_{n \neq 0} \frac{1}{n} \alpha_n^a \sin(n\sigma) e^{-in\tau}\,, \tag{4.17}$$

with reality condition $(\alpha_n^a)^* = \alpha_{-n}^a$ and Poisson brackets

$$\{\alpha_m^a, \alpha_n^b\} = -im\delta_{n+m}\delta^{a,b}\,. \tag{4.18}$$

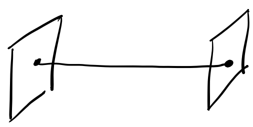
✎ **Exercise 4.4.** *Derive the mass formula in the ij sector*

$$\alpha' \hat{M}^2 = \left(\frac{x_i^2 - x_j^2}{2\pi\alpha'}\right)^2 + \frac{1}{\alpha'}(\hat{N} - 1) \tag{4.19}$$

with

$$\hat{N} := \sum_{n=1}^{\infty} \left(\left(\sum_{i=2}^{p} \hat{\alpha}_{-n}^i \hat{\alpha}_n^i \right) + \left(\sum_{a=p+1}^{D-1} \hat{\alpha}_{-n}^a \hat{\alpha}_n^a \right) \right)\,. \tag{4.20}$$

Note that as in the case of one D-brane, the mass formula in exercise 4.4 should be understood as the mass seen by a $(p+1)$-dimensional observer, since there is no momentum in the directions orthogonal to the D-brane. Note also that whenever $(x_i^a - x_j^a)^2 > 4\pi^2\alpha'$ the mass of the ground state is positive, while for $(x_i^a - x_j^a)^2 < 4\pi^2\alpha'$ it is tachyonic. A way of understanding this is that the string stretched between the two branes has a minimal tension, given by the stretching itself.



Going to the first excited states, they are of the form $\hat{\alpha}_{-1}^i |0\rangle$ and $\hat{\alpha}_{-1}^a |0\rangle$, with mass

$$\alpha' M^2 = \frac{(x_1^a - x_2^a)^2}{4\pi^2\alpha'}\,. \tag{4.21}$$

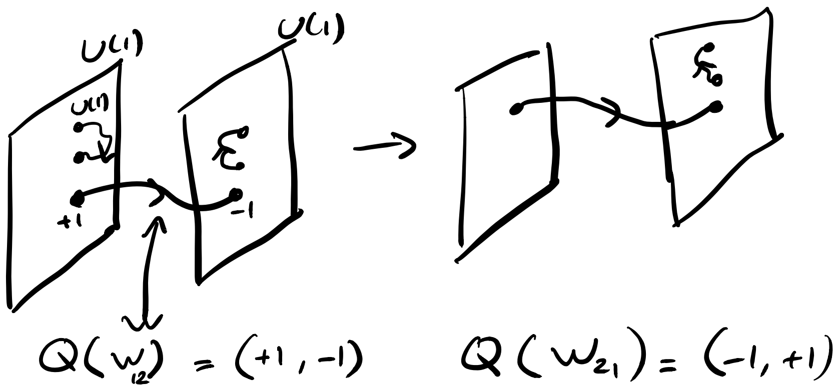
There is a small puzzle here: we have $p-1$ states of the form $\hat{\alpha}_{-1}^i |0\rangle$, which are massive for $x_1^a \neq x_2^a$. These are not enough to furnish a representation of the little group of massive particles in $p+1$ dimensions (that is, on the directions along the brane), which is $SO(p)$. What is happening here is that one of the massive scalars in $\hat{\alpha}_{-1}^a |0\rangle$ is "eaten" (by the Higgs mechanism) to make a full massive vector of the $(p+1)$-dimensional theory. So in this case we interpret the spectrum as a massive vector and $D-(p+2)$ massive scalars.

We have been discussing the 12 sector, but clearly the analysis in the 21 sector is totally identical, simply by exchanging $1 \leftrightarrow 2$. So we obtain another massive vector and $D-(p+2)$ scalars.

§4.3.1 Non-abelian enhancement

There is a natural question at this point: what happens if we now bring the two branes together? That is, what happens if we set $x_1^a = x_2^a$? The massive bosons that we found above become massless, so we have some additional gauge symmetry in the system. Which is the resulting gauge algebra?

The basic observation is that the 12 and 21 strings are not neutral under the $U(1)$ symmetries in the 11 and 22 sectors: we have that the 12 strings have charge $(+1, -1)$ under $U(1)_{11} \times U(1)_{22}$,[10] while the 21 strings have charge $(-1, +1)$. A somewhat heuristic argument, which can be made more precise, is that this charge arises because the endpoint of the 12 string can recombine with a suitable 11 string, to give a 12 string.



So a 12 string can "absorb" the $U(1)_{11}$ gauge bosons, and therefore carries a charge under $U(1)_{11}$. On the other hand a 12 and 21 string can recombine and make a closed string that leaves the brane system. This closed string is neutral under the gauge groups on the brane, so the 21 string must therefore have opposite charges to the 12 string.

This situation might be familiar from your studies of group theory, as it is the standard construction of the $U(2)$ in terms of a Cartan subalgebra (given by $U(1)_{11}$ and $U(1)_{22}$ in our case), and a set of roots. In this case the answer is that the resulting non-abelian group on the branes is $U(2)$. More generally, if we have N Dp-branes we have a $U(N)$ Yang-Mills theory in $p+1$ dimensions. As they arise in the same way as the gauge bosons, the scalars ϕ^a transform in the adjoint representation of this Yang-Mills theory. For a constant background $\langle\phi^a\rangle$ we can associate the eigenvalues of $\langle\phi^a\rangle$ with a displacement of the D-branes in the a-th trnasverse direction.

The case of the superstring is particularly interesting. In this case we have that N Dp-branes on top of each other give rise to *maximally supersymmetric* $U(N)$ Yang-Mills theory on $p+1$ dimensions. For instance, if $p = 3$ we have in this way four dimensional $\mathcal{N} = 4$ $U(N)$ SYM living on the branes.

[10] The fact that the two endpoints of the 12 string have opposite charges is purely conventional, because we can always redefine the signs of all $U(1)_{22}$ charges simultaneously as we wish, but it is a useful convention.

Let me close this discussion of D-branes with a couple of remarks. First, in order to fully specify a maximally supersymmetric Yang-Mills theory we need to give not only the gauge group, but also the gauge coupling g_{YM} of the field theory. As you might expect by now, this is not an extra parameter in the theory, but is rather determined by an expectation value of a dynamical field. In fact, this is an old friend: we have that (omitting some uninteresting normalisation factors) $g_{\mathrm{YM}}^2 \propto e^{\Phi}$, with Φ the closed string dilaton.

Recall from §3.6 that Φ also controlled the closed string perturbation theory. This brings me to my second remark: as I alluded to above, it is sometimes possible for open strings on the brane to recombine and emit a closed string. This implies that D-branes also couple to closed strings, and therefore to the (super)gravity background on which the closed strings move. In fact, D-branes were for many years understood only as rather mysterious objects in supergravity, quite analogous to black holes. It was only in 1995 that Polchinski realised (in hep-th/9510017) that these mysterious higher dimensional analogues of black holes could also be understood from the point of view of the string worldsheet as places where open strings could end. So D-branes admit a dual interpretation, both as open string endpoints, giving rise to Yang-Mills theories, and as highly curved backgrounds for the background supergravity. This dual nature of D-branes can be used to motivate Maldacena's AdS/CFT correspondence, see hep-th/9711200v3 for the original proposal.

§5 The superstring

§5.1 A supersymmetric Polyakov action

I will now introduce a variant of the construction above that avoids the problem of having a tachyon in the spacetime spectrum, known as the *superstring* (short for "supersymmetric string", or more precisely "string with supersymmetry on the worldsheet"). There are various interesting worldsheet theories that we can construct having supersymmetry in the worldsheet and supersymmetry in spacetime:

- Type IIA.
- Type IIB.
- Type I.
- Heterotic $SO(32)$.
- Heterotic $E_8 \times E_8$.

Here I will focus on the first two, known as the type II theories, as they are somewhat simpler to analyze with the tools that we have at hand. A natural question at this point is whether this list is complete, and even if it is, why should I impose supersymmetry. The answer to the first question is that this list is not complete: there are many more two dimensional theories that one can construct that have a graviton in their spectrum. But these are the only ones that we know with the Lorentz group $SO(1,9)$ as a global symmetry. All the other consistent constructions that we know have a Lorentz symmetry group $SO(1,d)$ with $d < 9$, and often can be understood as the result of placing one of the theories above on some $9-d$ dimensional manifold.

This leaves the other question: why should we impose supersymmetry on the worldsheet or target spacetime? The answer is simple but somewhat disappointing: theories without supersymmetry in either the worldsheet or the target exist, but without the aid of supersymmetry the analysis becomes too complicated and we typically can say very little about the resulting theory. For instance, it was realized early on that the IIA and IIB theories have some closely related partners known as the 0A and 0B theories, which have a very similar worldsheet structure (they are in particular supersymmetric), but do not preserve supersymmetry in the target. Much as in the case of the closed bosonic string, these theories have a tachyon in the closed string spectrum, and it is not easy to tell what happens upon tachyon condensation.[11]

Let me focus on the type II theories then. A systematic way of constructing and analyzing this theory would proceed along the same track that we followed for the classical

[11] Although in this case some rather sensible proposals exist, see for instance arXiv:hep-th/0012072, by Costa and Gutperle.

bosonic string in §2: start by defining a supersymmetric version of the Polyakov action, or in other words a two dimensional supergravity theory on the worldsheet, understand its symmetries, and then use these to choose a convenient gauge that simplifies the form of the action, making sure to remember to keep the equations of motion for the gauge fixed fields as constraints. This analysis is fairly straightforward, but also rather lengthy, so I will just outline it, only emphasizing those aspects that differ from the bosonic case. (You can find the details in a number of places, for example in the book by Blumenhagen, Lüst and Theisen.) The gauge fixed Polyakov action for the superstring is

$$S_P = -\frac{T}{2}\int_\Sigma d^2\boldsymbol{\sigma}\,\eta^{ab}\left(\partial_a X^\mu \partial_b X^\nu - i\bar{\psi}^\mu \rho_a \partial_b \psi^\nu\right)\eta_{\mu\nu}\,. \tag{5.1}$$

In addition to the embedding fields $X^\mu(\boldsymbol{\sigma})$, we now have a set of D worldsheet fermions $\psi^\mu(\boldsymbol{\sigma})$. We denote the two-dimensional Γ matrices by ρ^a, satisfying the Clifford algebra

$$\{\rho^a, \rho^b\} = -2\eta^{ab}\,. \tag{5.2}$$

(Note that $\{a, b\} = ab + ba$ denotes the anticommutator, not to be confused with the Poisson bracket $\{f, g\}_{\text{P.B.}}$ we have used above.) We will use the representation

$$\rho^0 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad ; \qquad \rho^1 = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}\,. \tag{5.3}$$

These matrices are purely imaginary, and thus the Lorentz group generators and $i\rho^a\partial_a$ purely real, so it makes sense to impose the Majorana reality condition

$$\psi := \begin{pmatrix} \psi_- \\ \psi_+ \end{pmatrix} \tag{5.4}$$

with $\psi_\pm^* = \psi_\pm$. This condition implies that $\psi^\dagger = \psi^t$, and thus $\bar{\psi} = \psi^t \rho^0$.

✎ **Exercise 5.1.** *Show that for Majorana spinors ψ, χ we have*

$$\bar{\psi}\chi = \bar{\chi}\psi \qquad \textit{and} \qquad \bar{\psi}\rho^a\chi = -\bar{\chi}\rho^a\psi\,. \tag{5.5}$$

Note that the two-dimensional chirality Γ-matrix $\bar{\rho} = \rho^0\rho^1$ has $\bar{\rho}\psi_\pm = \mp\psi_\pm$, so we can view ψ_+ and ψ_- as two independent Weyl fermions in the worldsheet. We see that in two dimensions having a defined chirality is compatible with the Majorana condition, so these are Majorana-Weyl fermions. (Such fermions exist in Lorentzian signature $(d-1, 1)$ only when $d \equiv 2 \bmod 8$.)

Note that the worldsheet spinors $\psi^\mu(\boldsymbol{\sigma})$ transform as vectors in spacetime. This is necessary in order for Lorentz invariance in the target space, which is a global symmetry from the point of view of the worldsheet, to commute with the supersymmetry generators in the worldsheet, which in this gauge act as

$$\delta X^\mu = \bar{\epsilon}\psi^\mu \tag{5.6a}$$
$$\delta\psi^\mu = -i\epsilon\rho^a\partial_a X^\mu \tag{5.6b}$$

where $\bar{\epsilon}$ is a constant spinor parameter in the worldsheet. This constant supersymmetry transformation is a remnant of the supergravity symmetries that we have gauged away.

✎ **Exercise 5.2.** *Show that the transformations (5.6) are symmetries of (5.1) for* $\epsilon(\boldsymbol{\sigma})$ *satisfying* $\rho^b \rho_a \partial_b \epsilon = 0$.

As in the analysis of the bosonic string, we still need to ensure that the equations of motion for the supergravity fields that we have gauged away. The first constraint is that the energy momentum tensor vanishes, coming from the equation of motion for the zweibein that we have gauged away:[12]

$$T_{ab} = \partial_a X^\mu \partial_b X_\mu + \frac{i}{4}\bar{\psi}^\mu \rho_a \partial_b \psi_\mu + \frac{i}{4}\bar{\psi}^\mu \rho_b \partial_a \psi_\mu - (\text{trace}) = 0 \tag{5.7a}$$

where the trace term is a term proportional to η_{ab} that imposes, as in the case of the bosonic string, that $\eta^{ab}T_{ab} = 0$, which should still be satisfied identically due to Weyl invariance of the action.

We have a supersymmetric theory, so we should expect that there is a femionic analogue of the constraint $T_{ab} = 0$, and indeed there is. It reads

$$J_a = \frac{1}{2}\rho^b \rho_a \psi^\mu \partial_b X_\mu = 0\,. \tag{5.7b}$$

This fermionic current can be derived as the Noether current associated to the supersymmetry transformations (5.6), or in terms of the original supergravity theory (that we have not described) as the equation of motion for the gravitino field that we have gauge fixed away to obtain the simpler form (5.1).

✎ **Exercise 5.3.** *Show, by direct computation, that, similarly to* $\eta^{ab}T_{ab} = 0$*, we have* $\rho^a J_a = 0$ *without having to use the equations of motion.*

[12] It is somewhat subtle to generalize our definition (2.5) of the energy-momentum tensor to a theory with fermions. The right way to do so is to write the metric in terms of zweibein fields, and then define T_{ab} to be proportional to the variation of the metric with respect to the zwebein, see for instance §8.3 in the book "Supergravity" by Freedman and van Proeyen. A simpler method to obtain (5.7a) is to recall that T_{ab} is also the Noether current associated to constant translations on the worldsheet.

§5.2 Left and right movers

The equation of motion for ψ is the Dirac equation

$$\rho^a \partial_a \psi^\mu = 0 \tag{5.8}$$

which in components reads

$$\begin{pmatrix} 0 & i(-\partial_0 + \partial_1) \\ i(\partial_0 + \partial_1) & 0 \end{pmatrix} \begin{pmatrix} \psi_-^\mu \\ \psi_+^\mu \end{pmatrix} = 0\,. \tag{5.9}$$

We can equivalently write this as

$$\partial_- \psi_+^\mu = 0\,, \tag{5.10a}$$
$$\partial_+ \psi_-^\mu = 0\,. \tag{5.10b}$$

The general solution of this equations is then that $\psi_+^\mu = \psi_+^\mu(\sigma^+)$ is purely left moving, and that $\psi_-^\mu = \psi_-^\mu(\sigma^-)$ is right moving. This is also clear if we write the fermionic part of the action in terms of the $+-$ coordinates:

$$S_F = iT \int_\Sigma d^2\boldsymbol{\sigma} (\psi_- \cdot \partial_+ \psi_- + \psi_+ \cdot \partial_- \psi_+)\,. \tag{5.11}$$

This implies, in particular, that it is consistent at this level to keep only one of the Majorana-Weyl fermions, either ψ_+ or ψ_-. This is the crucial observation that leads to the heterotic strings, but in here we will proceed along the somewhat simpler path of keeping both right movers and left movers.

It is useful, as always, to express the constraints in terms of $+-$ coordinates. Note first that for $\rho^\pm = \rho^0 \pm \rho^1$ we have $(\rho^\pm)^2 = 0$. Using this fact, a short calculation shows that the conditions $J_\pm = 0$ are equivalent to the two conditions

$$\psi_+^\mu \partial_+ X_\mu = 0\,, \tag{5.12a}$$
$$\psi_-^\mu \partial_- X_\mu = 0\,. \tag{5.12b}$$

Similarly, the $T_{ab} = 0$ constraints can be written in this basis as

$$T_{++} = \partial_+ X^\mu \partial_+ X^\nu \eta_{\mu\nu} - \frac{i}{2} \psi_+^\mu \partial_+ \psi_+^\nu \eta_{\mu\nu} \tag{5.13a}$$
$$T_{--} = \partial_- X^\mu \partial_- X^\nu \eta_{\mu\nu} - \frac{i}{2} \psi_-^\mu \partial_- \psi_-^\nu \eta_{\mu\nu} \tag{5.13b}$$

with $T_{+-} = T_{-+} = 0$ due to the tracelessness condition.

§5.3 Boundary conditions and mode expansions

The bosonic degrees of freedom X^μ in the superstring behave exactly as in the bosonic string, since they completely decouple from the ψ^μ in (5.1). We now construct the mode expansion of the fermionic modes.

Open string

We start by considering the open string. The variation of (5.11) picks up a boundary term

$$\delta S_F = \frac{iT}{2}\int d\tau (\psi_- \cdot \delta\psi_- - \psi_+ \cdot \delta\psi_+)\Big|_{\sigma=0}^{\sigma=\pi} . \tag{5.14}$$

In order for this term to vanish for all τ we impose that $\psi_+ = \pm\psi_-$ at each end of the open string.[13] It is always possible, by redefining the sign of ψ_-^μ if necessary, to choose $\psi_+(\tau,0) = \psi_-(\tau,0)$, but once we do this the choice of sign at $\sigma = \pi$ is physical. Let me treat the two options separately.

Ramond (R) boundary conditions (open string). Assume first that we choose $\psi_+(\tau,\pi) = +\psi_-(\tau,\pi)$. This choice of boundary conditions is known as *Ramond* (or "R") boundary conditions. (Note that we use a different typography for this in order to distinguish from the *"R"* label we use for right movers.) We have the mode expansion

$$\psi_-^\mu(\sigma^-) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma^-} \tag{5.15a}$$

$$\psi_+^\mu(\sigma^+) = \frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}} d_n^\mu e^{-in\sigma^+} . \tag{5.15b}$$

The Majorana condition $(\psi_\pm^\mu)^* = \psi_\pm^\mu$ then requires $(d_n^\mu)^* = d_{-n}^\mu$ for the (Grassmann) coefficients in the expansion.

Neveu-Schwarz (NS) boundary conditions (open string). The other possible choice is $\psi_+(\tau,\pi) = -\psi_-(\tau,\pi)$, known as the *Neveu-Schwarz* (or "NS") boundary condition. The mode expansion in this case is

$$\psi_-^\mu(\sigma^-) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma^-} \tag{5.16a}$$

$$\psi_+^\mu(\sigma^+) = \frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}} b_r^\mu e^{-ir\sigma^+} \tag{5.16b}$$

[13] What would happen if we impose $\delta\psi_\pm = 0$ or $\psi_\pm = 0$ instead? Because $\psi_\pm$ depends on τ via $\sigma^\pm$, this would freeze the value of $\psi_\pm$ everywhere, so this amounts to not introducing $\psi_\pm$ in the first place.

where now the sum is over half-integers $(\ldots,-\frac{1}{2},\frac{1}{2},\frac{3}{2},\ldots)$. The Majorana condition on ψ again requires $(b^\mu_r)^*=b^\mu_{-r}$.

Closed string

In the case of the closed string we need to impose that the fermions are compatible with the periodicity $\sigma\sim\sigma+\pi$ of the worldsheet. There are two choices for the left movers: it could be that $\psi^\mu_+(\sigma^+)=\psi^\mu_+(\sigma^++\pi)$, or that $\psi^\mu_+(\sigma^+)=-\psi^\mu_+(\sigma^++\pi)$. That is, we allow the fermions to be either periodic ("Ramond", or "R") or antiperiodic ("Neveu-Schwarz", or "NS") as we go around the spatial direction.[14] Similarly, there are two independent choices for the right movers, for a total of four independent choices. We denote these choices R-R, NS-R, R-NS and NS-NS. For instance, R-NS means that left movers are periodic and right movers antiperiodic.

Ramond boundary conditions (closed string). In analogy with the case of the open string, we call the periodic boundary conditions "Ramond", or simply "R". The mode expansion for right movers of Ramond type is given by

$$\psi^\mu_-(\sigma^-)=\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}d^\mu_n e^{-2in\sigma^-} \tag{5.17}$$

and the mode expansion for Ramond left movers is similarly

$$\psi^\mu_+(\sigma^+)=\frac{1}{\sqrt{2}}\sum_{n\in\mathbb{Z}}\tilde{d}^\mu_n e^{-2in\sigma^+}\,. \tag{5.18}$$

I emphasize that in this case d^μ_n and $\tilde{d}^\mu_n$ are independent sets of oscillators. The Majorana condition is $(d^\mu_n)^*=d^\mu_{-n}$ and $(\tilde{d}^\mu_n)^*=\tilde{d}^\mu_{-n}$.

Neveu-Schwarz boundary conditions (closed string). We call the antiperiodic boundary conditions "Neveu-Schwarz", or simply "NS". The mode expansions are

$$\psi^\mu_-(\sigma^-)=\frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}}b^\mu_r e^{-2ir\sigma^-} \tag{5.19}$$

for the right movers and

$$\psi^\mu_+(\sigma^+)=\frac{1}{\sqrt{2}}\sum_{r\in\mathbb{Z}+\frac{1}{2}}\tilde{b}^\mu_r e^{-2ir\sigma^+} \tag{5.20}$$

for the left movers, with $(b^\mu_r)^*=b^\mu_{-r}$ and $(\tilde{b}^\mu_r)^*=\tilde{b}^\mu_{-r}$ as usual.

[14] An obvious question at this point is why didn't we allow ourselves the same freedom for the bosonic coordinates X^μ. The answer is that we may, as long as we gauge the symmetry $X^\mu\sim -X^\mu$. (In introducing the NS sector we are implicitly saying that we are gauging $(-1)^F$, see the discussion in §5.5.) This is known as *orbifolding*, and it is a very interesting operation to consider. But it breaks Poincaré invariance in the target spacetime, which is why we have implicitly discarded this possibility in our analysis so far.

§5.4 *Quantization and the spacetime spectrum*

Other than the fact that we are dealing with perhaps slightly unfamiliar anticommuting fields, it is fairly straightforward to perform canonical quantization of the classical theory that we have just described. The result is that the oscillator modes are promoted to quantum operators with commutation relations reproducing the Poisson brackets.

The closed string

The commutation relations for the closed string in the bosonic sector are just as in (3.2), which I reproduce here for convenience:

$$[\hat{\alpha}^\mu_m, \hat{\alpha}^\nu_n] = [\hat{\tilde{\alpha}}^\mu_m, \hat{\tilde{\alpha}}^\nu_n] = m\delta_{m+n}\eta^{\mu\nu}\,, \tag{5.21a}$$

$$[\hat{x}^\mu, \hat{p}^\nu] = i\eta^{\mu\nu}\,. \tag{5.21b}$$

We also have fermionic operators, which commute with all bosonic commutators, and have (anti)commutators

$$\{\hat{b}^\mu_r, \hat{b}^\nu_s\} = \delta_{r+s}\eta^{\mu\nu} \qquad \text{(NS)} \tag{5.22a}$$

$$\{\hat{d}^\mu_m, \hat{d}^\nu_n\} = \delta_{m+n}\eta^{\mu\nu} \qquad \text{(R)} \tag{5.22b}$$

in the right moving sector of the closed string and similarly, with tildes, in the left moving sector of the closed strings.

The reality conditions on X^μ and ψ^μ imply that

$$(\alpha^\mu_m)^\dagger = \alpha^\mu_{-m} \qquad (b^\mu_r)^\dagger = b^\mu_{-r} \qquad (d^\mu_m)^\dagger = d^\mu_{-m} \tag{5.23}$$

and similarly for the left-moving oscillators.

Clearly, the same issue with ghosts (negative norm states) that we had in the closed string appears here, but it can be dealt with similarly by going to lightcone variables. We still have the unfixed bosonic transformations (3.8) that allow us to set

$$X^+ = p^+\tau \tag{5.24}$$

but additionally we have the unfixed local supersymmetry transformations in exercise (5.2), which allow us to set[15]

$$\psi^+_\pm = 0 \tag{5.25}$$

or equivalently $b^+_r = 0$, $d^+_m = 0$, $\tilde{b}^+_r = 0$ and $\tilde{d}^+_m = 0$ depending on which sector we are in. As in the bosonic case, we can now use the constraint equations (5.12) and (5.13) to

[15]This is in fact only possible in the NS sector. In the R sector we need to keep the zero modes $\hat{d}^+_0$ and $\hat{\tilde{d}}^+_0$. These modes will not play an important role in what follows, so we will ignore this subtlety.

solve for the oscillators in the X^- direction in terms of the oscillators in the transverse directions, with the result

$$\psi_\pm^- = \frac{1}{p^+}\psi_\pm^i \partial_\pm X_i\,, \tag{5.26a}$$

$$\partial_\pm X^- = \frac{1}{p^+}\left(\partial_\pm X^i \partial_\pm X^i + \frac{i}{2}\psi_\pm^i \partial_\pm \psi_\pm^i\right)\,. \tag{5.26b}$$

These equations determine X^- and ψ^- completely, up to a constant term x^- in X^-.

✎ **Exercise 5.4.** *Work out the form of* (5.26) *in terms of the oscillator modes.*

The remaining oscillators in the transverse directions are physical, and satisfy the algebra

$$\{\hat{b}_r^i, \hat{b}_s^j\} = \{\hat{\tilde{b}}_r^i, \hat{\tilde{b}}_s^j\} = \delta_{r+s}\delta^{i,j} \tag{5.27a}$$

$$\{\hat{d}_m^i, \hat{d}_n^j\} = \{\hat{\tilde{d}}_m^i, \hat{\tilde{d}}_n^j\} = \delta_{m+n}\delta^{i,j} \tag{5.27b}$$

$$[\hat{\alpha}_m^i, \hat{\alpha}_n^j] = [\hat{\tilde{\alpha}}_m^i, \hat{\tilde{\alpha}}_n^j] = m\delta_{m+n}\delta^{i,j}\,, \tag{5.27c}$$

$$[\hat{x}^i, \hat{p}^j] = i\delta^{i,j}\,, \tag{5.27d}$$

$$[\hat{x}^-, \hat{p}^+] = -i\,. \tag{5.27e}$$

Any commutators between transverse modes not appearing here vanishes.

In order to write the mass formula conveniently, we introduce the bosonic number operators

$$\hat{N}_{R,B} \coloneqq \sum_{n=1}^{\infty}\sum_{i=2}^{D-1} \hat{\alpha}_{-n}^i \hat{\alpha}_n^i \tag{5.28a}$$

$$\hat{N}_{L,B} \coloneqq \sum_{n=1}^{\infty}\sum_{i=2}^{D-1} \hat{\tilde{\alpha}}_{-n}^i \hat{\tilde{\alpha}}_n^i\,. \tag{5.28b}$$

The Neveu-Schwarz sector (closed string). We define a right moving NS number operator

$$\hat{N}_{R,\mathsf{NS}} \coloneqq \sum_{i=2}^{D-1}\sum_{r=\frac{1}{2}}^{\infty} r\hat{b}_{-r}^i \hat{b}_r^i \tag{5.29a}$$

and similarly for the left movers

$$\hat{N}_{L,\mathsf{NS}} \coloneqq \sum_{i=2}^{D-1}\sum_{r=\frac{1}{2}}^{\infty} r\hat{\tilde{b}}_{-r}^i \hat{\tilde{b}}_r^i \tag{5.29b}$$

where the sum runs over $r = \frac{1}{2}, \frac{3}{2}, \ldots$ in both cases.

Recall from §3.3 that the right moving bosonic oscillators contribute to the mass an amount

$$\begin{aligned}
\alpha' \hat{M}^2_{R,B} &= 2\left(\sum_{i=2}^{D-1} \frac{1}{2} \sum_{n\neq 0} \hat{\alpha}^i_{-n}\hat{\alpha}^i_n\right) \\
&= 2\left(\sum_{i=2}^{D-1} \frac{1}{2} \sum_{n=1}^{\infty} (\hat{\alpha}^i_n\hat{\alpha}^i_{-n} + \hat{\alpha}^i_{-n}\hat{\alpha}^i_n)\right) \\
&= 2\left(\hat{N}_{R,B} + \frac{D-2}{2}\zeta(-1)\right) \\
&= 2\left(\hat{N}_{R,B} - \frac{D-2}{24}\right).
\end{aligned} \tag{5.30}$$

It is possible to adapt the heuristic argument we gave in §3.3 so that it applies to the fermions in the NS sector. We postulate that the right mass operator in the right moving NS fermionic sector is of the form

$$\begin{aligned}
\alpha' \hat{M}^2_{R,\mathsf{NS},F} &= 2\left(\sum_{i=2}^{D-1} \frac{1}{2} \sum_{r\in\mathbb{Z}+\frac{1}{2}} r\hat{b}^i_{-r}\hat{b}^i_r\right) \\
&= 2\left(\sum_{i=2}^{D-1} \frac{1}{2} \sum_{r=\frac{1}{2}}^{\infty} r(\hat{b}^i_{-r}\hat{b}^i_r - \hat{b}^i_r\hat{b}^i_{-r})\right) \\
&= 2\left(\hat{N}_{R,\mathsf{NS}} - \frac{D-2}{2}\left(\frac{1}{2}+\frac{3}{2}+\frac{5}{2}+\ldots\right)\right)
\end{aligned} \tag{5.31}$$

We can define the sum as follows:

$$\begin{aligned}
\frac{1}{2}+\frac{3}{2}+\frac{5}{2}+\ldots &= \frac{1}{2}(1+3+5+\ldots) \\
&= \frac{1}{2}((1+2+3+4+5+\ldots)-(2+4+6+8+\ldots)) \\
&= \frac{1}{2}(\zeta(-1)-2\zeta(-1)) \\
&= -\frac{\zeta(-1)}{2}.
\end{aligned} \tag{5.32}$$

(This is probably a good time to remind the reader that the same constants can be obtained more rigorously by repeating our argument in §3.4 for the generators of the Lorentz group (taking into account the fermionic contributions). As before, $[J^{i-}, J^{j-}] = 0$ only vanishes for some specific choices of the critical dimension and relation between the classical and quantum constraints, which reproduce the values obtained from the heuristic manipulations above. Details of the derivation are given in §4.3 of the book by Green, Schwarz and Witten, for instance.)

Putting the fermionic and bosonic contributions together, we find out that the mass of right moving states in the NS sector is then given by

$$\begin{aligned}\alpha' \hat{M}^2_{R,\mathsf{NS}} &= \alpha'(\hat{M}^2_{R,B} + \hat{M}^2_{R,\mathsf{NS},F}) \\ &= 2\left(\hat{N}_{R,B} + \hat{N}_{R,\mathsf{NS}} + \frac{3(D-2)}{4}\zeta(-1)\right) \\ &= 2\left(\hat{N}_{R,B} + \hat{N}_{R,\mathsf{NS}} - \frac{D-2}{16}\right) .\end{aligned} \tag{5.33}$$

We will see below that the spectrum in the NS-NS sector includes the degrees of freedom of a massless graviton (among other fields), so Lorentz invariance (in the sense that states assemble in representations of the little group, as we imposed for the bosonic string) will require a critical dimension

$$\boxed{D = 10} \tag{5.34}$$

for the superstring. This implies that the mass contribution from the right movers, if they are in the NS sector, becomes

$$\boxed{\alpha' \hat{M}^2_{R,\mathsf{NS}} = 2\left(\hat{N}_{R,B} + \hat{N}_{R,\mathsf{NS}} - \frac{1}{2}\right) .} \tag{5.35}$$

We define the right moving NS vacuum as the state with lowest contribution to the spacetime mass, namely the one satisfying

$$\hat{\alpha}^i_n \left|0\right\rangle_{\mathsf{NS}} = \hat{b}^i_r \left|0\right\rangle_{\mathsf{NS}} = 0 \tag{5.36}$$

for all $i \in \{2, \ldots, 9\}$, $n > 0$ and $r > 0$, and similarly in the left moving sector.

The Ramond sector (closed string). The quantization of the R sector is more involved. The periodic boundary condition is compatible with supersymmetry, and this leads to a cancellation between the contribution of bosons and fermions to the formula for the spacetime mass:

$$\boxed{\alpha' \hat{M}^2_{R,\mathsf{R}} = 2\left(\hat{N}_{R,B} + \hat{N}_{R,\mathsf{R}}\right)} \tag{5.37}$$

with

$$\hat{N}_{R,\mathsf{R}} = \sum_{i=2}^{9} \sum_{n=1}^{\infty} n \hat{d}^i_{-n} \hat{d}^i_n \tag{5.38}$$

and similarly for the left movers:

$$\hat{N}_{L,\mathsf{R}} = \sum_{i=2}^{9} \sum_{n=1}^{\infty} n \hat{\tilde{d}}^i_{-n} \hat{\tilde{d}}^i_n \tag{5.39}$$

Note that $[\hat{d}^i_0, \hat{N}_{R,\mathsf{R}}] = 0$, so acting with the $\hat{d}^i_0$ operators does not increase the spacetime mass of states. This means that the vacuum, defined as the set of lowest mass states, is degenerate in the R sector: there are multiple states with lowest mass. These states transform in a linear representation of the operators $\hat{d}^i_0$, which obey the algebra

$$\{\hat{d}^i_0, \hat{d}^j_0\} = \delta^{i,j} \,. \tag{5.40}$$

This implies that we cannot simply impose $\hat{d}^i_0 \left|0\right\rangle_{\mathsf{R}} = 0$. In fact, if we define $\gamma^i := \sqrt{2}\hat{d}^i_0$, the algebra satisfied by the γ^i is precisely the Clifford algebra in 8 Euclidean dimensions:

$$\{\gamma^i, \gamma^j\} = 2\delta^{i,j} \,. \tag{5.41}$$

It is a well known fact that every representation of the Clifford algebra decomposes into copies of a single (up to isomorphism) representation (sometimes known as the *Dirac* representation), which can be built as follows. Introduce the eight matrices ($a \in \{1, \ldots, 4\}$)

$$A^a_\pm := \frac{1}{2}(\gamma^{2a} \pm i\gamma^{2a+1}) \,. \tag{5.42}$$

These matrices satisfy

$$\{A^a_-, A^b_+\} = \delta^{a,b} \qquad ; \qquad \{A^a_+, A^b_+\} = \{A^a_-, A^b_-\} = 0 \,, \tag{5.43}$$

so we can meaningfully define a state $\left|0\right\rangle_{\mathsf{R}}$ by imposing $A^a_- \left|0\right\rangle_{\mathsf{R}} = 0$ for all a. The rest of states at zero mass can then be constructed explicitly as follows

$$\begin{array}{rr} \left|0\right\rangle_{\mathsf{R}} & A^{a_1}_+ \left|0\right\rangle_{\mathsf{R}} \\ A^{a_1}_+ A^{a_2}_+ \left|0\right\rangle_{\mathsf{R}} & A^{a_1}_+ A^{a_2}_+ A^{a_3}_+ \left|0\right\rangle_{\mathsf{R}} \\ A^1_+ A^2_+ A^3_+ A^4_+ \left|0\right\rangle_{\mathsf{R}} & \end{array}$$

These states provide an irreducible representation of the Clifford algebra in eight dimensions, so it is natural to expect that they transform as a spinor under the spacetime Lorentz group for the eight transverse directions. This is indeed true: one way to see this is to notice that the spacetime Lorentz generators J^{ij} contain a term proportional to $\frac{1}{2}[\hat{d}^i_0, \hat{d}^j_0] = [\gamma^i, \gamma^j]$, which is the standard way in which one constructs the spinorial representation of the Lorentz generators out of a representation of the Clifford algebra.

✎ **Exercise 5.5.** *Construct the mode expansion for the Lorentz generators $J^{\mu\nu}$ in the superstring.*

So we have found that the vacuum in the R sector transforms as a 16 component Dirac spinor of Spin(8). As a representation of the Spin group, this representation is reducible into two eight component Weyl representations of definite chirality under the chirality matrix $\Gamma = \gamma^2 \cdots \gamma^9$. Since $\{\Gamma, A_+^i\} = 0$, the Ramond vacuum splits into the two irreducible representations

$$|\mathbf{8_s}\rangle = \{|0\rangle_{\mathsf{R}}\,, A_+^{a_1} A_+^{a_2}\,|0\rangle_{\mathsf{R}}\,, A_+^1 A_+^2 A_+^3 A_+^4\,|0\rangle_{\mathsf{R}}\}\,, \tag{5.44a}$$

$$|\mathbf{8_c}\rangle = \{A_+^{a_1}\,|0\rangle_{\mathsf{R}}\,, A_+^{a_1} A_+^{a_2} A_+^{a_3}\,|0\rangle_{\mathsf{R}}\}\,. \tag{5.44b}$$

Level matching. Finally, we need to impose the constraints $\hat{L}_0 = \hat{\bar{L}}_0$, or in other words the level matching conditions. An argument entirely analogous to the one we used in the context of the bosonic string then shows that this constraint becomes

$$\boxed{\hat{M}^2_{R,\phi} = \hat{M}^2_{L,\phi'}\,,} \tag{5.45}$$

where $\hat{M}_{R,\phi}$ stands for either $\hat{M}_{R,\mathsf{R}}$ or $\hat{M}_{R,\mathsf{NS}}$, depending on which periodicity we have chosen for the right moving fermions, and similarly for the left moving sector. That is, the left and right movers must contribute equally to the mass of any physical state, regardless of whether they are in the Ramond or Neveu-Schwarz sectors.

A tachyon. We have now all of the tools that we need to construct the spacetime spectrum of the superstring. The lowest mass state is in the NS-NS sector; it is simply the NS vacuum for the left movers times the NS vacuum for the right movers. Its mass is

$$\alpha' \hat{M}^2 = \alpha' \hat{M}^2_{R,\mathsf{NS}} + \alpha' \hat{M}^2_{L,\mathsf{NS}} = 2\left(-\frac{1}{2} - \frac{1}{2}\right) = -2\,. \tag{5.46}$$

So the vacuum in the NS-NS sector is again a tachyon! Nevertheless, the situation is better than in the bosonic string, because as we have defined it the superstring is actually inconsistent, and in fixing the inconsistency via the *GSO projection* described below, we will also be able to remove the tachyon in the spectrum.

The open string

Before going into that, let me quickly write the mass formulas for the open string sector from completeness. In the Ramond sector we have

$$\alpha' \hat{M}^2_{\mathsf{R}} = \hat{N}_B + \hat{N}_{\mathsf{R}} \tag{5.47}$$

and in the Neveu-Schwarz sector

$$\alpha' \hat{M}^2_{\mathsf{NS}} = \hat{N}_B + \hat{N}_{\mathsf{NS}} - \frac{1}{2}\,. \tag{5.48}$$

§5.5 The GSO projection and the spectrum of the type II theories

In fact, in introducing the NS sector we introduced a potential source of inconsistency in the theory.[16] The reason that we have to be careful in introducing antiperiodic boundary conditions in the theory is that we are really in a theory of gravity in two dimensions, and there exist large diffeomorphisms of Σ that relate different choices of boundary conditions for the fermions. One way of phrase this in modern language is that the 2d theory on Σ is only consistent in the presence of NS sectors if we gauge $(-1)^F$, where F measures fermion number on the worldsheet.

In general, gauging a symmetry involves summing over all possible backgrounds for the symmetry when doing the integral. Since the symmetry that we are considering is $(-1)^F$, this sum over backgrounds is a sum over spin structures on Σ.[17] There are various consistent ways of taking this sum over backgrounds, but it would sadly take us too far afield to classify them.[18]. When all is said and done, the result of the analysis is that all these ways of taking the sum lead to projecting onto a subset of the states that we found above. The operators that implement these projections are known as *"GSO projections"*. There are various consistent choices for the projections that one takes in the spectrum, but here I will describe the two that lead to spacetime supersymmetry and no tachyon in the spectrum.

Let me introduce some definitions. I define the right moving worldsheet fermion number $(-1)^{F_R}$ operator on the NS sector to satisfy

$$(-1)^{F_R}\hat{b}^i_r = -\hat{b}^i_r(-1)^{F_R} \tag{5.49a}$$

$$(-1)^{F_R}\hat{\alpha}^i_n = \hat{\alpha}^i_n(-1)^{F_R} \tag{5.49b}$$

and declare that $(-1)^{F_R}\left|0\right\rangle_{\mathsf{NS}} = -\left|0\right\rangle_{\mathsf{NS}}$. Equivalently, we can define $(-1)^{F_R} = -(-1)^{\hat{N}_{R,\mathsf{NS}}}$. So the NS vacuum is "fermionic", and $(-1)^{F_R}$ measures how many fermionic operators have acted on the vacuum to construct the state of interest. We can define an operator $(-1)^{F_L}$ on a NS left moving sector analogously by $(-1)^{F_L} = -(-1)^{\hat{N}_{L,\mathsf{NS}}}$.

For the right moving R sector, we define

$$(-1)^{F_R} = \Gamma(-1)^{\hat{N}_{R,\mathsf{R}}} \tag{5.50}$$

with $\Gamma = 16d_0^2\cdots d_0^9$ as above. For the vacua in the R sector we then have

$$(-1)^{F_R}\left|\mathbf{8_s}\right\rangle = +\left|\mathbf{8_s}\right\rangle \tag{5.51a}$$

$$(-1)^{F_R}\left|\mathbf{8_c}\right\rangle = -\left|\mathbf{8_c}\right\rangle \tag{5.51b}$$

[16] As we will see, the graviton lives in the NS-NS sector, so we need to introduce this sector if we want to obtain gravitons in the spacetime spectrum of the string.

[17] The original paper that discussed this viewpoint for the GSO projection is the paper by Seiberg and Witten called Spin Structures in String Theory. See arXiv:1911.11780 for a more modern take on the same perspective.

[18] It is nevertheless a rather beautiful story, with connections to algebraic topology and condensed matter physics. See the last reference in footnote 17 if you are interested in the details.

and each operator d_n^i introduces an additional minus sign. The operator $(-1)^{F_L}$ can be defined entirely analogously in the left moving sector.

We are now in the position to introduce the GSO projections that lead to the IIA and IIB theories.

The IIA theory

For the IIA theory we only keep states with $(-1)^{F_R} = (-1)^{F_L} = +1$ in the NS sector, and $(-1)^{F_R} = -(-1)^{F_L} = 1$ on the R sector (note the sign). For instance, this implies that the NS-NS vacuum, which recall was tachyonic, is projected out, since it is odd under $(-1)^{F_R}$ and $(-1)^{F_L}$. The states at the massless sector that survive the projection are easy to compute, and they correspond to the spectrum of the non-chiral $\mathcal{N} = (1, 1)$ supergravity in ten dimensions. In detail, the spectrum is as follows:

NS-NS sector. States of the form

$$|ij\rangle = \left(\hat{\tilde{b}}^i_{-\frac{1}{2}} |0\rangle_{\mathsf{NS}}\right) \otimes \left(\hat{b}^j_{-\frac{1}{2}} |0\rangle_{\mathsf{NS}}\right) \tag{5.52}$$

are massless and satisfy level matching. These states have $(-1)^{F_R} = (-1)^{F_L} = +1$, so they survive the GSO projection. Under the little group $SO(8)$ they transform in the $\mathbf{8} \otimes \mathbf{8}$ representation, which, exactly as in the case of the bosonic string, includes a graviton $G_{\mu\nu}$, an antisymmetric two-form $B_{\mu\nu}$, and the dilaton Φ.

R-R sector. The massless states that survive the IIA GSO projection in the R-R sector are of the form

$$|\mathbf{8_c}\rangle \otimes |\mathbf{8_s}\rangle \ . \tag{5.53}$$

In terms of the little group $SO(8)$, the product of two spinors of opposite chiralities decomposes into $\mathbf{8_v}$ and $\mathbf{56_v}$. This is a massless vector, associated to a $U(1)$ gauge boson in spacetime, known as C_1, and a massless three-form, which we generally call C_3.

R-NS sector. The massless states in this sector arise from

$$|\mathbf{8_c}\rangle \otimes \left(\hat{b}^i_{-\frac{1}{2}} |0\rangle_{\mathsf{NS}}\right) \tag{5.54}$$

In terms of the little group these transform as a gravitino and a spinor (known as the dilatino) of the same chirality, $\mathbf{8_s}$ and $\mathbf{56_s}$.

NS-R sector. In this case the massless states arise from

$$\left(\hat{\tilde{b}}^i_{-\frac{1}{2}} |0\rangle_{\mathsf{NS}}\right) \otimes |\mathbf{8_s}\rangle \ . \tag{5.55}$$

We again get a spinor and a gravitino, but this time they are of opposite chirality to the ones in the R-NS sector, being in the $\mathbf{8_c}$ and $\mathbf{56_c}$ representations.

The IIB theory

The GSO projection in the IIB theory is that $(-1)^{F_R} = (-1)^{F_L} = 1$ in both NS and R sectors. This again projects out the tachyon. At the massless level we find the spectrum of the chiral $\mathcal{N} = (0,2)$ supergravity in ten dimensions, as follows:

NS-NS sector. This is identical to the IIA case, with the states

$$|ij\rangle = \left(\hat{b}^i_{-\frac{1}{2}}\,|0\rangle_{\mathsf{NS}}\right) \otimes \left(\hat{\tilde{b}}^j_{-\frac{1}{2}}\,|0\rangle_{\mathsf{NS}}\right) \tag{5.56}$$

giving rise to a graviton, a 2-form and a dilaton.

R-R sector. The invariant states now come from

$$|\mathbf{8_s}\rangle \otimes |\mathbf{8_s}\rangle \tag{5.57}$$

which decomposes into representations of the massless little group $SO(8)$ as a four form C_4 subject to the self-duality condition $dC_4 = \star dC_4$, a two-form $C_{\mu\nu}$ and a scalar C_0 known as the "axion".[19]

R-NS sector. The massless spectrum are now

$$|\mathbf{8_s}\rangle \otimes \left(\hat{b}^i_{-\frac{1}{2}}\,|0\rangle_{\mathsf{NS}}\right) . \tag{5.58}$$

These states are a dilatino and gravitino in the $\mathbf{8_c}$ and $\mathbf{56_c}$ representations.

NS-R sector. This case works as in IIA. The massless states are

$$\left(\hat{\tilde{b}}^i_{-\frac{1}{2}}\,|0\rangle_{\mathsf{NS}}\right) \otimes |\mathbf{8_s}\rangle \tag{5.59}$$

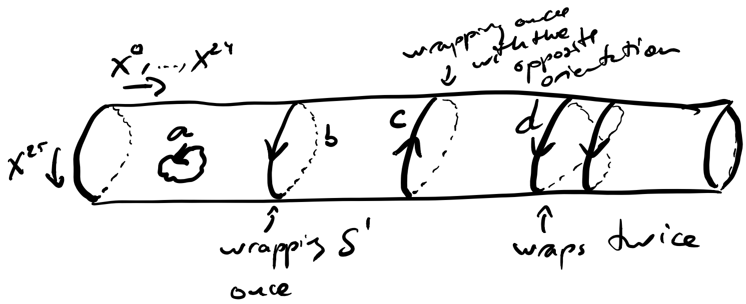
and transform as a gravitino and dilatino of the same chirality as those in the R-NS sector, namely $\mathbf{8_c}$ and $\mathbf{56_c}$.

[19] Although we do not need it in analysing the IIA or IIB theories, let me list for completeness the decomposition of $|\mathbf{8_c}\rangle \otimes |\mathbf{8_c}\rangle$. This gives a four form $\tilde{C}_4$ obeying $d\tilde{C}_4 = - \star d\tilde{C}_4$, a two-form $\tilde{C}_{\mu\nu}$ and a scalar $\tilde{C}_0$.

§6 T-duality

§6.1 Compactification and T-duality

So far we have considered the case of strings propagating on $\mathbb{R}^{1,D-1}$. We now see what happens in the case of strings propagating on $\mathbb{R}^{1,D-2} \times S^1$, with $X^{25} \sim X^{25} + 2\pi R$. Since one of the coordinates is periodic, closed strings can "wind" around the periodic direction a number of times, as in the following figure:



That is, we have different possible winding sectors. In terms of the wordsheet fields, the four winding sectors shown in the picture obey:

$$X_a^{25}(\tau, \sigma + \pi) = X_a^{25}(\tau, \sigma)\,, \tag{6.1a}$$

$$X_b^{25}(\tau, \sigma + \pi) = X_b^{25}(\tau, \sigma) + 2\pi R\,, \tag{6.1b}$$

$$X_c^{25}(\tau, \sigma + \pi) = X_c^{25}(\tau, \sigma) - 2\pi R\,, \tag{6.1c}$$

$$X_d^{25}(\tau, \sigma + \pi) = X_d^{25}(\tau, \sigma) + 4\pi R\,. \tag{6.1d}$$

Clearly we have infinite possibilities, which motivates the following definition.

Definition 6.1. A closed string with *winding number* $m \in \mathbb{Z}$ along a compact direction with radius R, say $X^{25} \sim X^{25} + 2\pi R$, satisfies

$$X^{25}(\tau, \sigma) = X^{25}(\tau, \sigma) + (2\pi R)m\,. \tag{6.2}$$

More abstractly, winding numbers are classified by the homotopy class of maps from the periodic space direction in the string worldsheet, with topology S^1_σ, to the target spacetime, which in the current case is $\mathbb{R}^{1,D-2} \times S^1$. These are classified by

$$\pi_1(\mathbb{R}^{1,D-2} \times S^1) = \pi_1(S^1) = \mathbb{Z}\,. \tag{6.3}$$

As usual, let us consider the mode expansion of the string in this background. The mode expansion for the X^i coordinates with $i < 25$ is unchanged, so we focus on the mode expansion for X^{25}. We still have a solution in terms of left and right movers:

$$X^{25}(\tau, \sigma) = X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-) \tag{6.4}$$

obeying the boundary condition (6.2), which in terms of left and right movers becomes:

$$X_L^{25}(\sigma^+ + \pi) - X_L^{25}(\sigma^+) = -X_R(\sigma^- - \pi) + X_R(\sigma^-) + 2\pi R m\,. \tag{6.5}$$

Note that the left hand side of this equality is a function depending on σ^+, while the right hand side is a function depending on σ^-. These are independent variables, so the only way that these two functions can be equal is if they are equal to some constant. If we take derivatives the constant disappears, and we obtain

$$(X_L^{25})'(\sigma^+ + \pi) = (X_L^{25})'(\sigma^+)\,, \tag{6.6a}$$
$$(X_R^{25})'(\sigma^+ - \pi) = (X_R^{25})'(\sigma^+)\,. \tag{6.6b}$$

Therefore $(X_{L,R}^{25})'$ are periodic with period π, so they admit Fourier expansions of the form

$$(X_L^{25})'(\sigma^+) = \ell \sum_{n\in\mathbb{Z}} \tilde{\alpha}_n^{25} e^{-2in\sigma^+}\,, \tag{6.7a}$$
$$(X_R^{25})'(\sigma^-) = \ell \sum_{n\in\mathbb{Z}} \alpha_n^{25} e^{-2in\sigma^-}\,. \tag{6.7b}$$

Integrating, this gives:

$$X_L^{25}(\sigma^+) = x_L^{25} + \frac{1}{2}\ell^2 p_L^{25}\sigma^+ + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25} e^{-2in\sigma^+}\,, \tag{6.8a}$$
$$X_R^{25}(\sigma^-) = x_R^{25} + \frac{1}{2}\ell^2 p_R^{25}\sigma^- + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25} e^{-2in\sigma^-}\,, \tag{6.8b}$$

with $p_L^{25} := 2\tilde{\alpha}_0^{25}/\ell$ and $p_R^{25} := 2\alpha_0^{25}/\ell$. So far the analysis is identical to the one for the ordinary string (compare with (2.21)), but we now encounter some differences. Note that we have

$$X_L^{25}(\sigma^+ + \pi) - X_L^{25}(\sigma^+) = \ell\tilde{\alpha}_0^{25}\pi \tag{6.9a}$$
$$-X_R^{25}(\sigma^- - \pi) + X_R^{25}(\sigma^+) = \ell\alpha_0^{25}\pi \tag{6.9b}$$

so the boundary condition (6.5) becomes

$$\ell\tilde{\alpha}_0^{25}\pi = \ell\alpha_0^{25}\pi + (2\pi R)m \tag{6.10}$$

or equivalently

$$\tilde{\alpha}_0^{25} - \alpha_0^{25} = \frac{2R}{\ell}m\,. \tag{6.11}$$

There is a subtler phenomenon that occurs whenever we have a compact dimension. Recall that the operator generating $X^{25} \to X^{25} + a$ translations in quantum mechanics is $\hat{T}(a) := e^{i\hat{p}^{25}a}$. In our case, because the X^{25} coordinate is periodic with period $2\pi R$, it must be the case that $\hat{T}(2\pi R)$ is a trivial operator. That is:

$$\hat{T}(2\pi R) = e^{i2\pi R\hat{p}^{25}} = 1\,. \tag{6.12}$$

So the eigenvalues of $\hat{p}^{25}$ should all be of the form k/R, with $k \in \mathbb{Z}$. Repeating the analysis in §2.4 we find:

$$p^{25} = p_L^{25} + p_R^{25}\,, \tag{6.13}$$

so we have that

$$\alpha_0^{25} + \tilde{\alpha}_0^{25} = \frac{\ell}{R}k\,. \tag{6.14}$$

We can therefore write:

$$p_L^{25} = \frac{k}{R} + \frac{Rm}{\alpha'}\,, \tag{6.15a}$$

$$p_R^{25} = \frac{k}{R} - \frac{Rm}{\alpha'}\,, \tag{6.15b}$$

so the contribution from the left and right movers to the momentum is no longer equal whenever we are in a sector with non-trivial winding number m.

Mass shell and level matching conditions

The classical expression for the right moving Virasoro generators is

$$L_m = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(-\alpha^+_{m-n}\alpha^-_n - \alpha^-_{m-n}\alpha^+_n + \alpha^{25}_{m-n}\alpha^{25}_n + \sum_{i=2}^{24}\alpha^i_{m-n}\alpha^i_n\right) \tag{6.16}$$

so in particular

$$L_0 = \frac{1}{2}\sum_{n=-\infty}^{\infty}\left(-\alpha^+_{-n}\alpha^-_n - \alpha^-_{-n}\alpha^+_n + \alpha^{25}_{-n}\alpha^{25}_n + \sum_{i=2}^{24}\alpha^i_{-n}\alpha^i_n\right) = \frac{1}{8}r_p^2 + N_i + N_{25} \tag{6.17}$$

where we have defined

$$r_p := (p^+, p^-, p^2, \ldots, p^{24}, p_R) \tag{6.18a}$$

$$N_i := \sum_{i=2}^{24}\sum_{n=1}^{\infty}\alpha^i_{-n}\alpha^i_n \tag{6.18b}$$

$$N_{25} := \sum_{n=1}^{\infty}\alpha^{25}_{-n}\alpha^{25}_n\,. \tag{6.18c}$$

Where r_p^2 denotes $r_p^\mu r_p^\nu \eta_{\mu\nu}$, as usual.

Now, because the momentum in the X^{25} direction is quantized, exciting any non-constant modes in this direction takes a finite amount of energy. So an observer without access to sufficient energy will see a theory that is effectively 25-dimensional, with an infinite tower of massive modes labelled by integers k. This is the idea of *Kaluza-Klein compactification*: a higher dimensional theory placed on a compact space appears, at low energies, to have fewer dimensions.[20]

So let us consider the situation as seen by a 25-dimensional observer. In order to do this, we write $r_p^2 = p_{25}^2 + p_R^2$, where $p_{25} := (p^+, p^-, p^2, \ldots, p^{24})$ is the momentum in the non-compact directions. The mass formula in 25 dimensions then follows from $L_0 = 0$

$$M^2 = -p_{25}^2 = \frac{8}{\ell^2}[N_i + N_{25}] + p_R^2\,. \tag{6.19}$$

We can also obtain the mass by looking to the left movers (that is, we impose $\tilde{L}_0 = 0$), we have

$$M^2 = -p_{25}^2 = \frac{8}{\ell^2}[\tilde{N}_i + \tilde{N}_{25}] + p_L^2 \tag{6.20}$$

with

$$\tilde{N}_i := \sum_{i=2}^{24}\sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^i \tilde{\alpha}_n^i \tag{6.21a}$$

$$\tilde{N}_{25} := \sum_{n=1}^{\infty} \tilde{\alpha}_{-n}^{25} \tilde{\alpha}_n^{25}\,. \tag{6.21b}$$

Imposing that both expressions for the mass agree gives the level matching condition

$$(N_i + N_{25}) - (\tilde{N}_i + \tilde{N}_{25}) = \frac{\ell^2}{8}\left(p_L^2 - p_R^2\right) = mk\,. \tag{6.22}$$

Notice in particular that whenever there is both momentum and winding in the compact direction the level of left and right movers differs.

We can take the average of the expression for the mass given by the left movers and the right movers to obtain a more symmetric expression:

$$\alpha' M^2 = 2\left(N_i + N_{25} + \tilde{N}_i + \tilde{N}_{25}\right) + \alpha'\left[\left(\frac{Rm}{\alpha'}\right)^2 + \left(\frac{k}{R}\right)^2\right]\,. \tag{6.23}$$

[20] So, for instance, superstring theory could in principle describe our four dimensional world if placed on some appropriate compact six-dimensional manifold. In practice, it is necessary to introduce, beyond the purely geometrical background, D-branes and other ingredients in the mix to get closer to the real world. We refer to such a system as a *compactification* of string theory. No compactification leading to the precise physics observed in our universe has been found yet, but steady progress in this direction has been made during the last two decades. A good introduction to the techniques and ideas involved in this endeavour is the book "String Theory and Particle Physics" by Ibañez and Uranga.

This was all for the classical theory. The quantization presents no additional difficulties compared to our analysis in §3. In the quantum theory we impose

$$\Phi_W(L_0) = \Phi_W(\tilde{L}_0) = 0 \tag{6.24}$$

where Φ_W is the Weyl quantization map introduced in §3.3. The result of the analysis is that level matching is modified to

$$(\hat{N}_i + \hat{N}_{25}) - (\hat{\tilde{N}}_i + \hat{\tilde{N}}_{25}) = mk \tag{6.25}$$

with

$$\hat{N}_i := \sum_{i=2}^{24}\sum_{n=1}^{\infty} \hat{\alpha}^i_{-n}\hat{\alpha}^i_n \tag{6.26a}$$

$$\hat{N}_{25} := \sum_{n=1}^{\infty} \hat{\alpha}^{25}_{-n}\hat{\alpha}^{25}_n \tag{6.26b}$$

and similarly for the left movers. The mass shell condition becomes (with the same regularization prescription $\sum_{n>0} n = -\frac{1}{12}$ as in §3.3)

$$\alpha'\hat{M}^2 = 2\left(\hat{N}_i + \hat{N}_{25} + \hat{\tilde{N}}_i + \hat{\tilde{N}}_{25} - 2\right) + \alpha'\left[\left(\frac{Rm}{\alpha'}\right)^2 + \left(\frac{k}{R}\right)^2\right]. \tag{6.27}$$

Note in particular the shift of -4 in $\alpha'\hat{M}^2$ with respect to the classical result.

T-duality

We now note that the formula (6.27) is invariant if we simultaneously exchange

$$m \leftrightarrow k \qquad \text{and} \qquad R \leftrightarrow \frac{\alpha'}{R}. \tag{6.28}$$

This is a rather surprising observation! We are saying that, at the level of the spectrum at least, there is no distinction between compactifying on a very large circle and compactifying on a very small circle. In fact, some reflection makes this equivalence somewhat reasonable: consider first the case of R small: then the modes with nontrivial momentum k become fairly large (this is natural: you need high energies to probe small distances), but winding modes become light. Heuristically, it becomes easy to wrap strings in the compact direction, but making a string move in the compact direction requires quite a bit of energy. The situation is precisely opposite when R is large: it is very easy to move in the compact dimension (in the limit $R \to \infty$ we go back to the 26 dimensional setting, where momentum is no longer discrete), but it is expensive to wrap a string.

In fact, the equivalence at the level of the spectrum can be promoted to an equivalence at the level of the full theory, known as *T-duality*. It is clear from the expressions for p_L^{25} and p_R^{25} in (6.15) that the action (6.28) leaving the spectrum invariant acts as $(p_L, p_R) \leftrightarrow (p_L, -p_R)$ on the left and right moving momenta. Our task is to show that this action on the momenta can be promoted to a symmetry of the full theory. Recall the mode expansion for the compactified theory that we found in (6.8), which we copy here for convenience:

$$X_L^{25}(\sigma^+) = x_L^{25} + \frac{1}{2}\ell^2 p_L^{25}\sigma^+ + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\tilde{\alpha}_n^{25}e^{-2in\sigma^+}\,, \tag{6.29a}$$

$$X_R^{25}(\sigma^-) = x_R^{25} + \frac{1}{2}\ell^2 p_R^{25}\sigma^- + \frac{i\ell}{2}\sum_{n\neq 0}\frac{1}{n}\alpha_n^{25}e^{-2in\sigma^-}\,. \tag{6.29b}$$

We can promote T-duality to a symmetry on the full left moving sector by simply declaring that it does nothing on x_L^{25} and the $\tilde{\alpha}_n^{25}$ oscillators. On the other hand the second term in X_R^{25} changes sign, so we cannot take a trivial action and keep the full theory invariant. Rather, we take the T-duality action on α_n^{25} to be $\alpha_n^{25} \leftrightarrow -\alpha_n^{25}$, and similarly $x_R^{25} \leftrightarrow -x_R^{25}$. A way of understanding better this second action is to write $x_L^{25} = \frac{1}{2}(x^{25} + q^{25})$ and $x_R^{25} = \frac{1}{2}(x^{25} - q^{25})$. The T-duality action is then $x^{25} \leftrightarrow q^{25}$. If we act in this way, we find that the putative T-duality invariance of the theory is

$$(X_L^{25}, X_R^{25}) \leftrightarrow (X_L^{25}, -X_R^{25})\,. \tag{6.30}$$

An overall sign change in X_R will lead to an isomorphic Hilbert space for the right movers, so the only potential concern comes from the Virasoro constraints, but it is clear from their expression (2.41) in terms of left and right movers that an overall sign change in X_R will not affect the form of these constraints.

So indeed, there is a full quantum equivalence for the bosonic string propagating on a circle of radius R and on a circle of radius α'/R. This is rather trivial from the worldsheet point of view, as we have seen, but it is a rather surprising behaviour from a target spacetime point of view! Let me emphasize that T-duality is only possible because we are dealing with strings, which are extended objects. Ordinary field theories with a finite number of fields do no exhibit T-duality.

T-duality for type II

The previous discussion was about the bosonic string, but the superstring also exhibits T-duality. Assume, as before, that we place either of the type II superstrings on $\mathbb{R}^{1,8} \times S_R^1$, with $X^9 \sim X^9 + 2\pi R$. The discussion in the bosonic sector is just as in the case of the bosonic string, so we have a potential duality acting as $(X_L^9, X_R^9) \leftrightarrow (X_L^9, -X_R^9)$. In order to preserve supersymmetry we therefore want

$$(\psi_L^9, \psi_R^9) \leftrightarrow (\psi_L^9, -\psi_R^9)\,. \tag{6.31}$$

So we find that the left moving sector of the superstring is entirely unaffected by T-duality. On the other hand the action on the right movers is more interesting.

The action on the NS is still not very interesting, as we just have a map

$$b^{25}_{-r}|0\rangle_{\mathsf{NS}} \leftrightarrow -b^{25}_{-r}|0\rangle_{\mathsf{NS}} \tag{6.32}$$

with $|0\rangle_{\mathsf{NS}}$ invariant. This is simply a relabelling of the states, but the theory is still isomorphic.

On the other hand, there is an interesting subtlety in the R sector. Recall from our discussion in §5.4 that the vacuum in the R sector is degenerate. We construct the vacuum by defining operators (5.42)

$$A^a_\pm := \frac{1}{2}(\gamma^{2a} \pm i\gamma^{2a+1}), \tag{6.33}$$

and then defining a state $|0\rangle_{\mathsf{R}}$ by imposing $A^a_-|0\rangle_{\mathsf{R}} = 0$ for all $a \in \{1,2,3,4\}$. The rest of the states in the lowest R sector are constructed by acting with A^a_+ on $|0\rangle_{\mathsf{R}}$. Finally, as in §5.5, we impose the GSO projection by demanding that in the physical theory we only keep states created by an even number of A^a_+ applications on the left moving R $|0\rangle_{\mathsf{R}}$ state (for the IIB string) or an odd number of A^a_+ application (for IIA).

Coming back to T-duality: the action (6.31) induces an action on the $A^a_\pm$ given by

$$(A^1_\pm, A^2_\pm, A^3_\pm, A^4_\pm) \leftrightarrow (A^1_\pm, A^2_\pm, A^3_\pm, A^4_\mp). \tag{6.34}$$

Note the change in sign in the last element, coming from the fact that $\gamma^9 = \sqrt{2}\hat{d}^9_0$ changes sign under T-duality. For clarity of argument, let me denote by $B^a_\pm$ the Clifford algebra operators in the T-dual picture, so that

$$(B^1_\pm, B^2_\pm, B^3_\pm, B^4_\pm) := (A^1_\pm, A^2_\pm, A^3_\pm, A^4_\mp). \tag{6.35}$$

If we start with a state $|0\rangle_{\mathsf{R}}$ in the original description, after T-duality it is no longer true that $B^4_-|0\rangle_{\mathsf{R}} = 0$. We instead define a new state $|0\rangle'_{\mathsf{R}} := B^4_-|0\rangle_{\mathsf{R}}$ such that $B^a_-|0\rangle'_{\mathsf{R}} = 0$ for all a. So far this is a minor relabelling of states without physical consequence, but it becomes rather significant when we apply the GSO projection: keeping the state $|0\rangle_{\mathsf{R}}$ in the spectrum would require imposing $(-1)^{F_R} = 1$ in the R sector in the original description, but $(-1)^{F_R} = -1$ in the new description (since $|0\rangle_{\mathsf{R}} = B^4_+|0\rangle'_{\mathsf{R}}$ in the T-dual description, which involves an odd number of B_+ applications).

Assume for concreteness that we start with the IIB theory on a circle of radius R. This involves the GSO projection $(-1)^{F_R} = (-1)^{F_L} = 1$ on both the NS and R sectors. As we have just argued, the T-dual theory on a circle of radius α'/R has identical projection on the left movers, and NS right movers, but projection $(-1)^{F_R} = -1$ on the R right movers. This is precisely the projection that defines the IIA theory![21] So, summarising:

$$\boxed{\text{IIA on } S^1_R = \text{IIB on } S^1_{\alpha'/R}\,.}$$

[21] In the conventions used in §5.5 this is only true up to an overall relabelling of what we mean by left and right movers, but this overall relabelling leads to a equivalent theory.

This is conceptually a very important result: while the IIA and IIB superstring theories look very different, they become exactly the same quantum theory once compactified on a circle. A slightly different perspective on this: if we start from the (unique) type II theory on a circle, the IIA and IIB ten dimensional theories arise from taking the $R \to 0$ and $R \to \infty$ limits.

§6.2 T-duality action on D-branes

There is a loose thread in the previous discussion that we need to tie up. We have shown that T-duality holds for the closed string. Does it also hold in the presence of D-branes? That is, does it also hold for open strings? Indeed it does, as we now show.

Consider the open bosonic string with NN boundary conditions in all directions, compactified on a circle of radius R, which we will take to be along the X^{25} direction. (It is not difficult to extend the discussion to the case of the superstring, one obtains the same result that we will obtain below.) We have

$$\partial_\sigma X^{25}(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0\,. \tag{6.36}$$

In terms of left and right movers this is

$$\partial_\sigma (X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-))\Big|_{\sigma=0,\pi} = 0\,. \tag{6.37}$$

Now, since $X_L^{25}(\sigma^+)$ and $X_R^{25}(\sigma^-)$ depend on σ only via $\sigma^\pm = \tau \pm \sigma$ we can use the chain rule to write

$$\partial_\sigma X_L(\sigma^+) = \partial_{\sigma^+} X_L^{25}(\sigma^+) = \partial_\tau X_L(\sigma^+) \tag{6.38a}$$
$$-\partial_\sigma X_R(\sigma^-) = \partial_{\sigma^-} X_R^{25}(\sigma^-) = \partial_\tau X_R(\sigma^-)\,, \tag{6.38b}$$

so we can rewrite the NN boundary condition as

$$\partial_\tau (X_L^{25}(\sigma^+) - X_R^{25}(\sigma^-))\Big|_{\sigma=0,\pi} = 0\,. \tag{6.39}$$

T-duality on the X^{25} direction sends $(X_L^{25}, X_R^{25}) \to (X_L^{25}, -X_R^{25})$, so in the T-dual description of the theory the boundary condition becomes

$$\partial_\tau (X_L^{25}(\sigma^+) + X_R^{25}(\sigma^-))\Big|_{\sigma=0,\pi} = \partial_\tau X^{25}(\tau,\sigma)\Big|_{\sigma=0,\pi} = 0\,, \tag{6.40}$$

which is the DD boundary condition that we have been writing as $\delta X^{25}(\tau,\sigma)\big|_{\sigma=0,\pi} = 0$.

So we learn that NN boundary conditions get mapped to DD boundary conditions. In the language of D-branes, we have that Dp-branes wrapping the S^1 with radius R turn into D$(p-1)$-branes localised at a point in the S^1 with radius α'/R.