Stochastic Control and Dynamic Asset Allocation, Partial lecture notes

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Notes on these notes

These notes were put together by G. d. R. for teaching SCDAA in 2015/16 and 2016/17. Parts of Chapter 1 and the exercises there were written by D. Š. for the RNAP course (hence there is a significant overlap). The notes are maintained and updated by D. Š. for the academic year 2017/18. The section on Dynamic Programming has been re-written. The section on BSDEs and Pontryagin's Maximum principle is also done differently to last year. Any mistakes, inaccuracies, confusing explanations in the current version are responsibility D. Š. Report any problems (e.g. write what the problem is, take a photo and email this).

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Contents

Reading these notes

Chapter 1 serves to introduce notation and some preliminary material. More preliminary material can be found in Appendix A. Most of the material there is covered by the DTF and SAF courses and you *must* be well familiar with the material there.

Chapter 2 is essential reading for what follows but Chapters 3 and 4 are basically independent of each other. There is another topic in the SCDAA course, which is the *Duality Theory* and this is not covered by these notes. This is again more or less independent from material in Chapters 3 and 4.

Exercises

You will find a number of exercises throughout these notes. You must make an effort to solve them (individually or with friends).

Solutions to some of the exercises will be made available as time goes by but remember: no one ever learned swimming by watching other people swim (and similarly no-one ever learned mathematics by reading others' solutions).

1 Background and Preparation

1.1 Basic notation and useful review of analysis concepts

Here we set the main notation for the rest of the course. These pages serve as an easy reference.

General For any two real numbers x, y,

$$x \wedge y = \min\{x, y\}, \quad x \vee y = \max\{x, y\}, \quad x^+ = \max\{x, 0\}, \quad x^- = \max\{-x, 0\}.$$

Sets, metrics and matrices $\mathbb{N}$ is the set of strictly positive integers and $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

$\mathbb{R}^d$ denotes the d-dimensional Euclidean space of real numbers. For any $x = (x_1, \cdots, x_d)$, $y = (y_1, \cdots, y_d)$ in $\mathbb{R}^d$, we denote the inner product by xy and by $|\cdot|$ the Euclidean norm i.e.

$$xy := \sum_{i=1}^{d} x_i y_i \quad \text{and} \quad |x| := \left(\sum_{i=1}^{d} x_i^2\right)^{\frac{1}{2}}$$

$\mathbb{R}^{d\times n}$ denotes the set of real valued $d \times n$-matrices; I_n denotes the $n \times n$-identity matrix. For any $\sigma \in \mathbb{R}^{n\times d}$, $\sigma = (\sigma_{ij})_{1\le i\le n, 1\le j\le d}$ we write the transpose of σ as $\sigma^\top = (\sigma_{ji})_{1\le j\le d, 1\le i\le n} \in \mathbb{R}^{d\times n}$. We write the trace operator of an $n \times n$-matrix σ as $\text{Tr}(\sigma) = \sum_{i=1}^n \sigma_{ii}$. For a matrices we will use the norm $|\sigma| := \big(\text{Tr}(\sigma\sigma^\top)\big)^{1/2}$.

Definition 1.1 (Supremum/Infimum)**.** Given a set $S \subset \mathbb{R}$, we say that μ is the supremum of S if *(i)* $\mu \ge x$ for each $x \in S$ and if *(ii)* for every $\varepsilon > 0$ there exists an element $y \in S$ such that $y > \mu - \varepsilon$. We write $\mu = \sup S$.

The infimum is defined symmetrically as follows: λ is the infimum if *(i)* $\lambda \le x$ for each $x \in S$ and if *(ii)* for every $\varepsilon > 0$ there exists an element $y \in S$ such that $y < \lambda + \varepsilon$. We write $\lambda = \inf S$.

Note that supremum is the *least upper bound*, i.e. the smallest real number greater than or equal to all the elements of the set S. Infimum is the *greatest lower bound*, i.e. the largest number smaller than or equal to all the elements of the set S. It is also important to note that the infimum (or supremum) do not necessarily have to belong to the set S.

Functions and functional spaces For any set A, the indicator function of A is

$$\mathbb{1}_A(x) = 1 \text{ if } x \in A, \qquad \text{otherwise } \mathbb{1}_A(x) = 0 \text{ if } x \notin A.$$

We write $C^k(A)$ is the space of all real-valued continuous functions on A with continuous derivatives up to order $k \in \mathbb{N}_0$, $A \subset \mathbb{R}^n$. In particular $C^0(A)$ is the space of real-valued functions on A that are continuous.

For a real-valued function functions $f = f(t, x)$ defined $I \times A$ we write $\partial_t f$, $\partial_{x_i} f$ and $\partial_{x_i x_j} f$ for $1 \le i, j \le n$ for its partial derivatives. By Df we denote the gradient vector of f and by $D^2 f$ the Hessian matrix of f (whose entries $1 \le i, j \le d$ are given by $\partial_{x_i x_j} f(t, x)$).

Consider an interval I (and think of I as a time interval $I = [0, T]$ or $I = [0, \infty)$). Then $C^{1,2}(I \times A)$ is the set of real valued functions $f = f(t, x)$ on $I \times A$ whose partial derivatives $\partial_t f$, $\partial_{x_i} f$ and $\partial_{x_i x_j} f$ for $1 \le i, j \le n$ exist and are continuous on $I \times A$.

Integration and probability We use $(\Omega, \mathcal{F}, \mathbb{P})$ to denote a probability space with $\mathbb{P}$ being the probability measure and $\mathcal{F}$ the σ-algebra.

"$\mathbb{P}$-a.s." denotes "almost surely for the probability measure $\mathbb{P}$" (we often omit the reference to $\mathbb{P}$). "μ-a.e." denotes "almost everywhere for the measure μ"; here μ will not be a probability measure. This means is that a statement Z made about $\omega \in \Omega$ holds $\mathbb{P}$-a.s. if there is a set $E \in \mathcal{F}$ such that $\mathbb{P}(E) = 0$ and Z is true for *all* $\omega \in E^c = \Omega \setminus E$.

$\mathcal{B}(U)$ is the Borel σ-algebra generated by the open sets of the topological space U.

$\mathbb{E}[X]$ is the expectation of the random variable X with respect to a probability $\mathbb{P}$. $\mathbb{E}[X|\mathcal{G}]$ is the conditional expectation of X given $\mathcal{G}$. The variance of the random variable X, possibly vector valued, is denoted by $\text{Var}(X) = \mathbb{E}[(X - \mathbb{E}(X))(X - \mathbb{E}(X))^\top]$.

Since we may define different measures on the same σ-algebra we must sometimes distinguish which measure is used for expectation, conditional expectation or variance. We thus sometimes write $\mathbb{E}^{\mathbb{Q}}[X]$, $\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]$ or $\text{Var}^{\mathbb{Q}}$ to show which measure was used.

1.1.1 General analysis definitions and inequalities

Definition 1.2 (Convex function)**.** A function $f : \mathbb{R} \to (-\infty, \infty]$ is called convex if

$$\forall\, \lambda \in [0,1]\ \forall\, x, y \in \mathbb{R} \qquad f\big(\lambda x + (1-\lambda)y\big) \le \lambda f(x) + (1-\lambda) f(y).$$

If a function f is convex then it is differentiable a.e. and (with f'_- denoting its left-derivative, f'_+ its right-derivative) and we have

$$f'_+(x) := \lim_{y \searrow x} \frac{f(y) - f(x)}{y - x} = \inf_{y > x} \frac{f(y) - f(x)}{y - x}\,,$$

$$f'_-(x) := \lim_{y \nearrow x} \frac{f(y) - f(x)}{y - x} = \sup_{y < x} \frac{f(y) - f(x)}{y - x}\,.$$

So, from the expression with infimum we see that,

if $y > x$ then $f'_+(x) \le \dfrac{f(y) - f(x)}{y - x}$ which implies $f(y) \ge f(x) + f'_+(x)(y - x)$ for $y > x$.

Moreover, from the expression with supremum we see that[1],

if $y < x$ then $f'_-(x) \ge \dfrac{f(y) - f(x)}{y - x}$ which implies $f(y) \ge f(x) + f'_-(x)(y - x)$ for $y < x$.

We review a few standard analysis inequalities, some not named and some others named: Cauchy-Schwarz, Holder, Young and Gronwall's inequality.

$$\begin{aligned} \forall\, x \in \mathbb{R} \qquad & x \le 1 + x^2 \\ \forall a, b \in \mathbb{R} \qquad & 2ab \le a^2 + b^2 \\ \forall\, n \in \mathbb{N}\ \forall a, b \in \mathbb{R} \qquad & |a + b|^n \le 2^{n-1}\big(|a|^n + |b|^n\big) \end{aligned}$$

[1] As $y < x$ we multiply by negative number, flipping the inequality.

Lemma 1.3 (Cauchy–Schwarz inequality). *Let H be a Hilbert space with inner product $(\cdot,\cdot)$ and norm $|\cdot|_H$. If $x,y\in H$ then $(x,y)\leq |x|_H|y|_H$.*

Example 1.4. i) If $x,y\in\mathbb{R}^d$ then $xy<|x||y|$.

ii) We can check that $L^2(\Omega)$ with inner product given by $\mathbb{E}[XY]$ for $X,Y\in L^2(\Omega)$ is a Hilbert space. Hence the Cauchy–Schwarz inequality is $\mathbb{E}[XY]\leq \big(\mathbb{E}[X^2]\big)^{1/2}\big(\mathbb{E}[Y^2]\big)^{1/2}$.

Lemma 1.5 (Young's inequality). *Let $a,b\in\mathbb{R}$. Then for any $\varepsilon\in(0,\infty)$ for any $p,q\in(1,\infty)$ such that $1/p+1/q=1$ it holds that*

$$ab\leq \varepsilon\frac{|a|^p}{p}+\frac{1}{\varepsilon}\frac{|b|^q}{q}\,.$$

The above inequality is not the original Young's inequality, that is for the choice $\varepsilon=1$. The one here is the original Young's inequality with the choice $(ab)=(\varepsilon a)(b/\varepsilon)$.

Lemma 1.6 (Gronwall's lemma / inequality). *Let $\lambda=\lambda(t)\geq 0$, $a=a(t)$, $b=b(t)$ and $y=y(t)$ be locally integrable, real valued functions defined on I (with $I=[0,T]$ or $I=[0,\infty)$) such that λy is also locally integrable and for almost all $t\in[0,T]$*

$$y(t)+a(t)\leq b(t)+\int_0^t\lambda(s)y(s)\,ds.$$

Then

$$y(t)+a(t)\leq b(t)+\int_0^t\lambda(s)e^{\int_s^t\lambda(r)dr}(b(s)-a(s))\,ds\quad\text{for almost all } t\in I.$$

Furthermore, if b is monotone increasing and a is non-negative, then

$$y(t)+a(t)\leq b(t)e^{\int_0^t\lambda(r)\,dr},\quad\text{for almost all } t\in I.$$

If the function y in Gronwall's lemma is continuous then the conclusions hold for all $t\in I$. For proof see Exercise 1.35.

1.1.2 Some fundamental probability results

(Following the notation established in SAF) we define $\liminf$ and $\limsup$.

Definition 1.7 (limsup & liminf). Let $(a_n)_{n\in\mathbb{N}}$ be any sequence in $\overline{\mathbb{R}}=\mathbb{R}\cup\{-\infty,\infty\}$

$$\begin{aligned}\liminf_{n\to\infty}a_n &:= \lim_{n\to\infty}\lim_{k\to\infty}\min\{a_n,a_{n+1},a_{n+2},\dots,a_k\}=\inf_n\sup_{k\geq n}a_k,\\ \limsup_{n\to\infty}a_n &:= \lim_{n\to\infty}\lim_{k\to\infty}\max\{a_n,a_{n+1},a_{n+2},\dots,a_k\}=\sup_n\inf_{k\geq n}a_k.\end{aligned}$$

Clearly $\liminf_{n\to\infty}a_n\leq\limsup_{n\to\infty}a_n$ and if $\lim_{n\to\infty}a_n=:a$ exists, then $\liminf_{n\to\infty}a_n=\limsup_{n\to\infty}a_n=a$. On the other hand, if $\liminf_{n\to\infty}a_n\geq\limsup_{n\to\infty}a_n$, then $\lim_{n\to\infty}a_n=a$ exists.

Exercise 1.8 ($\limsup$ and $\liminf$ of RV are RV). Show that $\liminf_{n\to\infty}X_n$ and $\limsup_{n\to\infty}X_n$ are random variables for any sequence of random variables X_n.

Lemma 1.9 (Fatou's lemma)**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence of non-negative random variables. Then*

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n].$$

Moreover,

i) If there exists a r.v. Y such that $\mathbb{E}[\,|Y|\,] < \infty$ and $Y \leq X_n$ $\forall n$ (allows $X_n < 0$), then

$$\mathbb{E}\left[\liminf_{n\to\infty} X_n\right] \leq \liminf_{n\to\infty} \mathbb{E}[X_n].$$

ii) If there exists a r.v. Y such that $\mathbb{E}[\,|Y|\,] < \infty$ and $Y \geq X_n$ $\forall n$, then

$$\mathbb{E}\left[\limsup_{n\to\infty} X_n\right] \geq \limsup_{n\to\infty} \mathbb{E}\left[X_n\right].$$

The first part of the above lemma does not require integrability of the sequence of $(X_n)_{n\in\mathbb{N}}$ due to the use of the Monotone Convergence Theorem in its proof. The enumerated statements follow as a corollary of the first statement. Of course, a version of Fatou's lemma using conditional expectations also exists (simply replace $\mathbb{E}[\cdot]$ with $\mathbb{E}[\cdot|\mathcal{F}_t]$).

Lemma 1.10 (Hölder's inequality)**.** *Let $(X, \mathcal{X}, \mu)$ be a measure space (i.e. X is a set, $\mathcal{X}$ a σ-algebra and μ a measure). Let $p, q > 1$ be real numbers s.t. $1/p + 1/q = 1$ or let $p = 1, q = \infty$. Let $f \in L^p(X, \mu)$, $g \in L^q(X, \mu)$. Then*

$$\int_X |fg|\, d\mu \leq \left(\int_X |f|^p d\mu\right)^{\frac{1}{p}} \left(\int_X |g|^q d\mu\right)^{\frac{1}{q}}$$

In particular if p, q are such that $1/p + 1/q = 1$ and $X \in L^p(\Omega)$, $Y \in L^q(\Omega)$ are random variables then

$$\mathbb{E}[\,|XY|\,] \leq \mathbb{E}[\,|X|^p\,]^{\frac{1}{p}} \mathbb{E}[\,|Y|^q\,]^{\frac{1}{q}}.$$

Lemma 1.11 (Minkowski's inequality or triangle inequality)**.** *Let $(X, \mathcal{X}, \mu)$ be a measure space (i.e. X is a set, $\mathcal{X}$ a σ-algebra and μ a measure). For any $p \in [1, \infty]$ and $f, g \in L^p(X, \mu)$*

$$\left(\int_X |f+g|^p\, d\mu\right)^{\frac{1}{p}} \leq \left(\int_X |f|^p\, d\mu\right)^{\frac{1}{p}} + \left(\int_X |g|^p\, d\mu\right)^{\frac{1}{p}}.$$

Lemma 1.12 (Jensen's inequality)**.** *Let f be a convex function and X be any random variable with $\mathbb{E}[|X|] < \infty$. Then*

$$f\big(\mathbb{E}[X]\big) \leq \mathbb{E}\big[f(x)\big].$$

1.2 Some useful results from stochastic analysis

For convenience we state some results from stochastic analysis. Proofs can be found for example in Stochastic Analysis for Finance lecture notes, in [Pha09], [Bjö09] or [KS91].

1.2.1 Probability Space

Let us always assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a fixed probability space. We assume that $\mathcal{F}$ is complete which means that all the subsets of sets with probability zero are included in $\mathcal{F}$. We assume there is a filtration $(\mathcal{F}_t)_{t\in[0,T]}$ (which means $\mathcal{F}_s \subseteq \mathcal{F}_t \subseteq \mathcal{F}$) such that $\mathcal{F}_0$ contains all the sets of probability zero.

1.2.2 Stochastic Processes, Martingales

A stochastic process $X = (X_t)_{t\geq 0}$ is a collection of random variables X_t which take values in $\mathbb{R}^d$.

We will always assume that stochastic processes are *measurable*. This means that $(\omega, t) \mapsto X(\omega)_t$ taken as a function from $\Omega \times [0,\infty)$ to $\mathbb{R}^d$ is measurable with respect to σ-algebra $\mathcal{F} \otimes \mathcal{B}([0,\infty))$. This product is defined as the σ-algebra generated by sets $E \times B$ such that $E \in \mathcal{F}$ and $B \in \mathcal{B}([0,\infty))$. From Theorem A.2 we then get that

$$t \mapsto X_t(\omega) \text{ is measurable for all } \omega \in \Omega.$$

We say X is $(\mathcal{F}_t)_{t\geq 0}$ *adapted* if for all $t \geq 0$ we have that X_t is $\mathcal{F}_t$-measurable.

Definition 1.13. Let X be a stochastic process that is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and such that for every $t \geq 0$ we have $\mathbb{E}[|X_t|] < \infty$. If for every $0 \leq s < t \leq T$ we have

i) $\mathbb{E}[X_t|\mathcal{F}_s] \geq X_s$ a.s.then the process is called *submartingale*.

ii) $\mathbb{E}[X_t|\mathcal{F}_s] \leq X_s$ a.s.then the process is called *supermartingale*.

iii) $\mathbb{E}[X_t|\mathcal{F}_s] = X_s$ a.s.then the process is called *martingale*.

For submartingales we have Doob's maximal inequality:

Theorem 1.14 (Doob's submartingale inequality)**.** *Let $X \geq 0$ be an $(\mathcal{F}_t)_{t\in[0,T]}$-submartingale and $p > 1$ be given. Assume $\mathbb{E}\left[X_T^p\right] < \infty$. Then*

$$\mathbb{E}\Big[\sup_{0\leq t\leq T} X_t^p\Big] \leq \left(\frac{p}{p-1}\right)^p \mathbb{E}\left[X_T^p\right].$$

Definition 1.15 (Local Martingale)**.** A stochastic process X is called a *local martingale* if is there exists a sequence of stopping time $(\tau_n)_{n\in\mathbb{N}}$ such that $\tau_n \leq \tau_{n+1}$ and $\tau_n \to \infty$ as $n \to \infty$ and if the *stopped process* $(X(t \wedge \tau_n))_{t\geq 0}$ is a martingale for every n.

Lemma 1.16 (Bounded from below local martingales are supermartingales)**.** *Let $(M_t)_{t\in[0,T]}$ be a local Martingale and assume it is positive or more generally bounded from below. Then M is a super-martingale.*

Proof. The proof makes use of Fatou's Lemma 1.9 above. Since M is a local Martingale then there exists a sequence of stopping times $(\tau_n)_{n\in\mathbb{N}}$ increasing to infinity a.s. such that the stopped process $M_t^n := M_{t\wedge\tau_n}$ is a Martingale. We have then, using Fatou's lemma for any $0 \leq s \leq t \leq T$

$$\mathbb{E}[M_t|\mathcal{F}_s] = \mathbb{E}[\liminf_{n\to\infty} M_t^n|\mathcal{F}_s] \leq \liminf_{n\to\infty} \mathbb{E}[M_t^n|\mathcal{F}_s] = \liminf_{n\to\infty} M_s^n = M_s,$$

and hence M is a supermartingale. □

Exercise 1.17 (Submartingale)**.** In view of the previous lemma, is a bounded from above local martingale a submartingale?

1.2.3 Integration Classes and Itô's Formula

Definition 1.18. By $\mathcal{H}$ we mean all $\mathbb{R}$-valued and adapted processes g such that for any $T > 0$ we have

$$\|g\|^2_{\mathcal{H}_T} := \mathbb{E}\left[\int_0^T |g_s|^2 ds\right] < \infty.$$

By $\mathcal{S}$ we mean all $\mathbb{R}$-valued and adapted processes g such that for any $T > 0$ we have

$$\mathbb{P}\left[\int_0^T |g_s|^2 ds < \infty\right] = 1.$$

Exercise 1.19. Show that $\mathcal{H} \subset \mathcal{S}$.

The importance of these two classes is that stochastic integral with respect to W is defined for all integrands in class $\mathcal{S}$ and this stochastic integral is a continous *local* martingale. For the class $\mathcal{H}$ the stochastic integral with respect to W is a martingale.

Definition 1.20. By $\mathcal{A}$ we denote $\mathbb{R}$-valued and adapted processes g such that for any $T > 0$ we have

$$\mathbb{P}\left[\int_0^T |g_s| ds < \infty\right] = 1.$$

By $\mathcal{H}^{d\times n}$, $\mathcal{S}^{d\times n}$ we denote processes taking values the space of $d\times n$-matrices such that each component of the matrix is in $\mathcal{H}$ or $\mathcal{S}$ respectively. By $\mathcal{A}^d$ we denote processes taking values in $\mathbb{R}^d$ such that each component is in $\mathcal{A}$

1.2.4 Itô processes and Itô Formula

We will need the multi-dimensional version of the Itô's formula. Let W be an n-dimensional Wiener martingale with respect to $(\mathcal{F})_{t\geq 0}$. Let $\sigma \in \mathcal{S}^{m\times d}$ and let $b \in \mathcal{A}^m$. We say that the d-dimensional process X has the stochastic differential

$$dX_t = b_t\, dt + \sigma_t\, dW_t \tag{1.1}$$

for $t \in [0, T]$, if

$$X_t = X_0 + \int_0^t b_s\, ds + \int_0^t \sigma_s\, dW(s).$$

Such a process is also called an *Itô process*.

The Itô formula or chain rule for stochastic processes Before we go into the main result, let us go over an example from classic analysis. Take three functions, $u = u(t, x)$, $g = g(t)$ and $h = h(t)$ given by $h(t) := u\big(t, g(t)\big)$. Let us compute $\frac{d}{dt}h(t)$.

Since h is given as a composition of functions, we use here is the standard chain for functions of several variables (this takes into account that the variation of h arising from changes in t comes from the variation of g and also from the first component in u). Thus we have

$$\frac{d}{dt}h(t) = \big(\partial_t u\big)\big(t, g(t)\big) + \big(\partial_x u\big)\big(t, g(t)\big)\frac{d}{dt}g(t).$$

We want to see the contrast with Itô formula, which has to be written in integral form (since W has almost everywhere non-differentiable paths). To that end, we integrate

$$\int_0^t \frac{d}{dt}h(s)\,ds = \int_0^t \big(\partial_t u\big)\big(s, g(s)\big)\,ds + \int_0^t \big(\partial_x u\big)\big(s, g(s)\big)\frac{d}{dt}g(s)\,ds$$

and use the Fundamental theorem of calculus

$$h(t) - h(0) = \int_0^t \big(\partial_t u\big)\big(s, g(s)\big)\,ds + \int_0^t \big(\partial_x u\big)\big(s, g(s)\big)\,dg(s)$$

which can be written in the differential notation as

$$dh(t) = \partial_t f\big(t, g(t)\big)\,dt + \partial_x f\big(t, g(t)\big)\,dg(t). \tag{1.2}$$

Compare (1.2) with (1.3) below. You see a fundamental difference: the second derivative term! It appears there exactly because the Wiener process has non-differentiable paths and hence a correction to (1.2) is needed.

We have then the following important result.

Theorem 1.21 (Multi-dimensional Itô formula). *Let X be a m-dimensional Itô process given by* (1.1). *Let $u \in C^{1,2}([0,T] \times \mathbb{R}^m)$. Then the process given by $u(t, X_t)$ has the stochastic differential*

$$\begin{aligned} du\big(t, X_t\big) = \partial_t u\big(t, X_t\big)\,dt &+ \sum_{i=1}^{d} \partial_{x_i} u\big(t, X_t\big)\,dX_t^i \\ &+ \frac{1}{2}\sum_{i,j=1}^{d} \partial_{x_i x_j} u\big(t, X_t\big)\,dX_t^i\,dX_t^j, \end{aligned} \tag{1.3}$$

where for $i, j = 1, \ldots, m$

$$dt\,dt = \,dt\,dW_t^i = 0, \quad dW_t^i\,dW_t^j = \delta_{ij}\,dt.$$

We now consider a very useful special case. Let X and Y be $\mathbb{R}$-valued Itô processes. We will apply to above theorem with $f(x, y) = xy$. Then $\partial_x f = y$, $\partial_y f = x$, $\partial_{xx} f = \partial_{yy} f = 0$ and $\partial_{xy} f = \partial_{yx} f = 1$. Hence from the multi-dimensional Itô formula we have

$$df\big(X_t, Y_t\big) = Y_t\,dX_t + X_t\,dY_t + \frac{1}{2}\,dY_t\,dX_t + \frac{1}{2}\,dX_t\,dY_t.$$

Hence we have the following corollary

Corollary 1.22 (Itô's product rule). *Let X and Y be $\mathbb{R}$-valued Itô processes. Then*

$$d\big(X_t Y_t\big) = X_t\,dY_t + Y_t\,dX_t + \,dX_t\,dY_t.$$

1.2.5 Martingale Representation Formula and Girsanov's theorem

Theorem 1.23 (Lévy characterization). *Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration. Let $X = (X_t)_{t\in[0,T]}$ be a continuous m-dimensional process adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ such that for $i = 1, \ldots, d$ the processes*

$$M_t^i := X_t^i - X_0^i$$

are local martingales with respect to $(\mathcal{F}_t)_{t\in[0,T]}$ and $dM_t^i\,dM_t^j = \delta_{ij}\,dt$ for $i, j = 1, \ldots, d$. Then X is a Wiener martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$.

So essentially any continuous local martingale with the right quadratic variation is a Wiener process.

Theorem 1.24 (Girsanov). *Let $(\mathcal{F}_t)_{t\in[0,T]}$ be a filtration. Let $W = (W_t)_{t\in[0,T]}$ be a d-dimensional Wiener martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$. Let $\varphi = (\varphi_t)_{t\in[0,T]}$ be a d-dimensional process adapted to $(\mathcal{F}_t)_{t\in[0,T]}$ such that*

$$\mathbb{E}\Big[\int_0^T |\varphi_s|^2\,ds\Big] < \infty.$$

Let

$$L_t := \exp\left\{-\int_0^t \varphi_s^\top\,dW(s) - \frac{1}{2}\int_0^t |\varphi_s|^2\,ds\right\} \tag{1.4}$$

and assume that $\mathbb{E}[L_T] = 1$. Let $\mathbb{Q}$ be a new measure on $\mathcal{F}_T$ given by the Radon-Nikodym derivative $d\mathbb{Q} = L(T)\,d\mathbb{P}$. Then

$$W_t^{\mathbb{Q}} := W_t + \int_0^t \varphi_s\,ds$$

is a $\mathbb{Q}$-Wiener martingale.

We don't give proof but only make some useful observations.

1. Clearly $L_0 = 1$.
2. The Novikov condition is a useful way of establishing that $\mathbb{E}[L_T] = 1$: if
$$\mathbb{E}\left[e^{\frac{1}{2}\int_0^T |\varphi_t|^2\,dt}\right] < \infty$$
then L is a martingale (and hence $\mathbb{E}[L_T] = \mathbb{E}[L_0] = 1$).
3. Applying Itô's formula to $f(x) = \exp(x)$ and
$$dX_t = -\varphi_t^\top\,dW_t - \frac{1}{2}|\varphi_t|^2\,dt$$
yields
$$dL_t = -L_t\varphi_t^\top\,dW_t.$$

Theorem 1.25 (Martingale representation). *Let $W = (W_t)_{t\in[0,T]}$ be a d-dimensional Wiener martingale and let $(\mathcal{F}_t)_{t\in[0,T]}$ be generated by W. Let $M = (M_t)_{t\in[0,T]}$ be a continuous real valued martingale with respect to $(\mathcal{F}_t)_{t\in[0,T]}$.*

Then there exists unique adapted d-dimensional process $h = (h_t)_{t\in[0,T]}$ such that for $t \in [0,T]$ we have

$$M_t = M_0 + \sum_{i=1}^d \int_0^t h_s^i\,dW_s^i.$$

If the martingale M is square integrable then h is in $\mathcal{H}$.

Essentially what the theorem is saying is that we can write continuous martingales as stochastic integrals with respect to some process as long as they're adapted to the filtration generated by the process.

1.3 Stochastic differential equations

In this chapter we review, without proofs, some standard results about stochastic differential equations and related partial differential equations.

1.3.1 Stochastic differential equations

On a given stochastic basis $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[0,\infty]}, \mathbb{P})$ we consider a stochastic differential equation (SDE) of the form,

$$dX_t = b(t, X_t)\, dt + \sigma(t, X_t)\, dW_t, \qquad X(0) = \xi, \tag{1.5}$$

for $t \in [0, T]$, if

$$X_t = X_0 + \int_0^t b(s, X_s)\, ds + \int_0^t \sigma(s, X_s)\, dW_s.$$

Here W is a d-dimensional Brownian motion, $\sigma(t, \cdot)$ is a $m \times d$-matrix, and $b(t, \cdot)$ is a m-dimensional vector. Written component-wise, the SDE reads

$$dX_t^i = b^i(t, X_t)\, dt + \sum_{j=1}^d \sigma^{ij}(t, X_t)\, dW_t^j, \quad X_0^i = \xi_i, \quad i \in \{1, \cdots, m\}.$$

The drift and volatility coefficients

$$(t, \omega, x) \mapsto \big(b(t, \omega, x), \sigma(t, \omega, x)\big)$$

are progressively measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}^d)$; as usual, we suppress ω in the notation and write $b(t, x)$ instead of $b(t, \omega, x)$. The initial value ξ is $\mathcal{F}_0$-measurable. Note that $t = 0$ plays no special role in this; we may as well start the SDE at some time $t \geq 0$ (even a stopping time), and we shall write $X^{t,x} = (X_s^{t,x})_{s\in[t,T]}$ for the solution of the SDE started at time t with initial value x (assuming it exists and is unique).

Remark 1.26. In the setup above the coefficients b and σ are random. In applications we will deal essentially with two settings for b and σ.

i) b and σ are deterministic (Borelian) functions, i.e. $b(t, x)$ and $\sigma(t, x)$ are not random. In this case, the corresponding SDE is called a *diffusion process*.

ii) b and σ are effectively random maps, but the randomness has a specific form. Namely, the random coefficients $b(t, \omega, x)$ and $\sigma(t, \omega, x)$ are of the form

$$b(t, \omega, x) := \widehat{b}\big(t, x, \alpha_t(\omega)\big) \quad \text{and} \quad \sigma(t, \omega, x) := \widehat{\sigma}\big(t, x, \alpha_t(\omega)\big)$$

where $\widehat{b}$, $\widehat{\sigma}$ are deterministic (Borelian) functions on $[0, T] \times \mathbb{R}^d \times U$, A is a complete separable metric space and $(\alpha_t)_{t\in[0,T]}$ is a progressively measurable process valued in A.

This case arises in stochastic control problems that we will study later on, an example of which can already be seen with SDE (2.1). This type of SDE, depending on a control process α is called a *controlled diffusion by* α.

Given $T \geq 0$, we write $\mathbb{H}_T^2$ for the set of progressively measurable processes ϕ such that

$$\|\phi\|_{\mathbb{H}_T^2} := \mathbb{E}\left[\int_0^T |\phi_t|^2\, dt\right]^{\frac{1}{2}} < \infty.$$

Proposition 1.27 (Existence and uniqueness of solutions)**.** *Let $\xi \in L^m(\mathcal{F}_0)$ and $T > 0$. Assume there exists a constant K such that*

$$\|b(\cdot,0)\|_{\mathbb{H}^2_T} + \|\sigma(\cdot,0)\|_{\mathbb{H}^2_T} \leq K,$$
$$|b(t,x) - b(t,y)| + |\sigma(t,x) - \sigma(t,y)| \leq K|x-y|, \qquad \mathbb{P}-a.s.$$

for all $t \leq T$ and $x, y \in \mathbb{R}^d$.

Then the SDE has a unique (strong) solution X on the interval $[0,T]$. Moreover, there exists a constant $C = C(K,T,m)$ such that

$$\mathbb{E}\left[\sup_{0\leq t\leq T} |X_t|^m\right] \leq C\big(1 + \mathbb{E}[\,|\xi|^m]\big).$$

1.3.2 General properties of SDEs

In the remainder, we always assume that the coefficients b and σ satisfy the above conditions.

Proposition 1.28 (Stability)**.** *Let $x, x' \in \mathbb{R}^m$ and $0 \leq t \leq t' \leq T$.*

i) There exists a constant $C = C(K,T,m)$ such that

$$\mathbb{E}\left[\sup_{t\leq s\leq T} |X_s^{t,x} - X_s^{t,x'}|^m\right] \leq C|x-x'|^m.$$

ii) Suppose in addition that there is a constant K' such that

$$\mathbb{E}\left[\int_s^{s'} |b(r,0)|^2 + |\sigma(r,0)|^2\,dr\right] \leq K'|s-s'|$$

for all $0 \leq s \leq s' \leq T$. Then there exists $C = C(K,T)$ such that

$$\mathbb{E}\left[\sup_{t'\leq s\leq T} |X_s^{t,x} - X_s^{t',x}|^2\right] \leq C(K + |x|^2)|t-t'|.$$

To prove the above two propositions one uses often the following inequalities: Cauchy-Schwartz, Holder and Young's inequality; Gronwall's inequality (see Lemma 1.6); Doob's maximal inequality (see Theorem 1.14).

Proposition 1.29 (Flow property)**.** *Let $x \in \mathbb{R}^m$ and $0 \leq t \leq t' \leq T$. Then*

$$X_s^{t,x} = X_s^{t',X_{t'}^{t,x}}, \qquad s \in [t',T].$$

(This property holds even if t, t' are stopping times.)

Proof. Exercise; deduce this property from the uniqueness of solutions of SDEs. □

Proposition 1.30 (Markov property)**.** *Let $x \in \mathbb{R}^d$ and $0 \le t \le t' \le s \le T$. If b and σ are* deterministic functions*, then*

$$X_s^{t,x} \quad \textit{is a function of } t,\ x,\ s,\ \textit{and } \big(W_r - W_t\big)_{r\in[t,s]}.$$

Moreover,

$$\mathbb{E}\left[\Phi\big(X_r^{t,x},\ t' \le r \le s\big)|\mathcal{F}_{t'}\right] = \mathbb{E}\left[\Phi\big(X_r^{t,x},\ t' \le r \le s\big)|X_{t'}^{t,x}\right]$$

for all bounded and measurable functions $\Phi : C^0([t', s]; \mathbb{R}^m) \to \mathbb{R}$.

On the left hand side (LHS), the conditional expectation is on $\mathcal{F}_{t'}$ that contains all the information from time $t = 0$ up to time $t = t'$. On the right hand side (RHS), that information is replaced by the process $X_{t'}^{t,x}$ at time $t = t'$. In words, for Markovian processes the best prediction of the future, given all knowledge of the present and past (what you see on the LHS), is the present (what you see on the RHS; all information on the past can be ignored).

1.3.3 PDEs and Feynman-Kac Formula

(This section can be traced back to either [Pha09] or SAF notes (Section 16).)

In the case of deterministic maps b and σ in (1.5), the so-called *diffusion SDE*, we can give the following definition of Infinitesimal generator.

Definition 1.31 (Infinitesimal generator (associated to an SDE))**.** Let b and σ be deterministic functions in (1.5). For all $t \in [0, T]$, the following second order differential operator $\mathcal{L}$ is called the *infinitesimal generator associated to the diffusion* (1.5),

$$\mathcal{L}\varphi(t, x) = b(t, x)D\varphi(t, x) + \frac{1}{2}\mathrm{Tr}(\sigma\sigma^\top D^2\varphi)(t, x), \qquad \varphi \in C^{0,2}([0, T] \times \mathbb{R}^m).$$

Although the above definition does seems weird and unfamiliar, the operator $\mathcal{L}$ appears every time one uses the Itô formula to $\varphi(t, X_t)$ where the process $(X_t)_{t\in[0,T]}$ is the solution to (1.5).

Exercise 1.32. Let $(X_t)_{t\in[0,T]}$ be the solution to (1.5).

Show that for $\varphi \in C^{1,2}([0, T] \times \mathbb{R})$, we have

$$d\varphi(t, X_t) = \big(\partial_t\varphi + \mathcal{L}\varphi\big)(t, X_t)\, dt + \big(\partial_x\varphi\sigma\big)(t, X_t)\, dW_t.$$

It is possible, for certain classes of SDE and differential equations, to write the solution to a PDE as an expectation of (a function of) the solution to the SDE associated to the differential operator appearing in the PDE; it is not surprising that the PDE differential operator must be the infinitesimal generator. That is the core message of the next result.

Theorem 1.33 (Feynman-Kac formula in 1-dim)**.** *Assume that the function $v : [0, T] \times \mathbb{R} \to \mathbb{R}$ belongs to $C^{1,2}([0, T) \times \mathbb{R}) \cap C^0([0, T] \times \mathbb{R})$ and is a solution to the following boundary value problem*

$$\partial_t v(t, x) + b(t, x)\partial_x v(t, x) + \frac{1}{2}\sigma^2(t, x)\partial_{xx}v(t, x) - rv(t, x) = 0, \tag{1.6}$$

$$v(T, x) = h(x), \tag{1.7}$$

where b and σ are deterministic functions.

For any $(t,x) \in [0,T] \times \mathbb{R}$, define the stochastic process $(X_s)_{s\in[t,T]}$ as the solution to the SDE

$$dX_s = b(s, X_s)\, ds + \sigma(s, X_s)\, dW_s, \ \ \forall\, s \in [t,T], \qquad X_t = x. \tag{1.8}$$

Assume that the stochastic process $\big(e^{-rs}\sigma(s,X_s)\partial_x v(s,X_s)\big)_{s\in[t,T]} \in L^2([0,T]\times\mathbb{R})$.

Then the solution v of (1.6)-(1.7) can be expressed as (with $\mathbb{E}_{t,x}[\cdot] = \mathbb{E}[\cdot|X_t = x]$)

$$v(t,x) = e^{-r(T-t)}\mathbb{E}_{t,x}\big[\, h\big(X_T\big)\,\big] \qquad \forall\,(t,x) \in [0, T \times \mathbb{R}.$$

Proof. The proof is rather straightforward and is based on a direct application of Itô's formula.

Define the process $(Y_s)_{s\in[t,T]}$ as $Y_s = e^{-rs}v(s,X_s)$ where X is given by (1.8). Applying Itô's formula to Y, i.e. computing dY_s gives

$$\begin{aligned}
dY_s =&\ d\Big(e^{-rs}v(s,X_s)\Big)\\
&= (-r)e^{-rs}v\, ds + e^{-rs}\partial_s v\, ds + e^{-rs}\partial_x v\, dX_s + \frac{1}{2}e^{-rs}\partial_{xx}v\big(\, dX_s\big)^2\\
&= e^{-rs}\big[-rv + \partial_t v + b\partial_x v + \frac{1}{2}\sigma^2\partial_{xx}v\big]\, ds + e^{-rs}\big[\sigma\partial_x v\big]\, dW_s,
\end{aligned}$$

where the v function is evaluated in point (s,X_s). Using the equality given by (1.6) we see that the ds term disappears completely leaving

$$dY_s =\ d\Big(e^{-rs}v(s,X_s)\Big) = e^{-rs}\big[\ \sigma(s,X_s)\partial_x v(s,X_s)\ \big]\, dW_s.$$

Integrating both sides from $s = t$ to $s = T$ gives

$$\begin{aligned}
\big[e^{-rs}v(s,X_s)\big]\Big|_{s=t}^{s=T} &= \int_t^T e^{-ru}\sigma(u,X_u)\partial_x v(u,X_u)\, dW_u\\
\Leftrightarrow e^{-rt}v(t,X_t) &= e^{-rT}v(T,X_T) - \int_t^T e^{-ru}\sigma(u,X_u)\partial_x v(u,X_u)\, dW_u,\\
\Leftrightarrow v(t,X_t) &= e^{-r(T-t)}v(T,X_T) - \int_t^T e^{-r(u-t)}\sigma(u,X_u)\partial_x v(u,X_u)\, dW_u.
\end{aligned}$$

Taking expectations $\mathbb{E}_{(t,x)}[\cdot]$ on both sides (recall that the process X starts at time t in position x; this is the meaning of the subscript (t,x) in the expectation sign),

$$v(t,X_t) = e^{-r(T-t)}\mathbb{E}_{t,x}\big[v(T,X_T)\big] = e^{-r(T-t)}\mathbb{E}_{t,x}\big[h(X_T)\big],$$

where the expectation of the stochastic integral disappears due to the properties of the stochastic integral, since by assumption we have $\big(e^{-rs}\sigma(s,X_s)\partial_x v(s,X_s)\big)_{s\in[t,T]} \in L^2([0,T]\times\mathbb{R})$. $\square$

Exercise 1.34 (Two extensions of the Feynman-Kac formula). a) Redo the previous proof when the constant r is replaced by a function $r : [0,T]\times\mathbb{R} \to \mathbb{R}$; assume r to be bounded and continuous. Hint instead of e^{-rs}, use $\exp\{-\int_t^s r(u,X_u)\, du\}$.

b) Redo the previous proof when the PDE (1.6) is equal to some $f(t,x)$ instead of being equal to zero.

1.4 Exercises

Exercise 1.35 (On Gronwall's lemma)**.** Prove Gronwall's Lemma by following these steps:

i) Let
$$z(t) = \left(e^{-\int_0^t \lambda(r)dr}\right) \int_0^t \lambda(s)y(s)\,ds.$$
and show that
$$z'(t) \leq \lambda(t)e^{-\int_0^t \lambda(r)dr}\left(b(t) - a(t)\right).$$

ii) Integrate from 0 to t to obtain the first conclusion Lemma 1.6.

iii) Obtain the second conclusion of Lemma 1.6.

Exercise 1.36 (On liminf)**.** Let $(a_n)_{n\in\mathbb{N}}$ be a bounded sequence. Then the number
$$\lim_{n\to\infty}\left(\inf\{a_k : k \geq n\}\right)$$
is called *limit inferior* and is denoted by $\liminf_{n\to\infty} a_n$.

1. Show that the limit inferior is well defined, that is, the limit $\lim_{n\to\infty}\left(\inf\{a_k : k \geq n\}\right)$ exists and is finite for any bounded sequence (a_n).

2. Show that the sequence $(a_n)_{n\in\mathbb{N}}$ has a subsequence that converges to $\lim_{n\to\infty} \inf a_n$.

   Hint: Argue that for any $n \in \mathbb{N}$ one can find $i \geq n$ such that
   $$\inf\{a_k : k \geq n\} \leq a_i < \inf\{a_k : k \geq n\} + \frac{1}{n}.$$
   Use this to construct the subsequence we are looking for.

Exercise 1.37 (Property of the supremum/infimum)**.** Let $a, b \in \mathbb{R}$. Prove that
$$\text{if } b > 0\text{, then} \quad \sup_{x\in X}\left\{a + bf(x)\right\} = a + b\sup_{x\in X} f(x),$$
$$\text{if } b < 0\text{, then} \quad \sup_{x\in X}\left\{a + bf(x)\right\} = a + b\inf_{x\in X} f(x).$$

The below exercises were created by Dr. David Siska and are the first set of exercises to the Course *Risk Neutral Asset Pricing (RNAP)*.

Exercise 1.38 (ODEs)**.** Assume that (r_t) is an adapted stochastic process such that for any $t \geq 0$ $\int_0^t |r_s|\,ds < \infty$ holds $\mathbb{P}$-almost surely (in other words $r \in \mathcal{A}$).

1. Solve
$$dB_t = B_t r_t dt, \quad B_0 = 1. \tag{1.9}$$

2. Is the function $t \mapsto B_t$ continuous? Why?

3. Calculate $d(1/B_t)$.

Exercise 1.39 (Geometric Brownian motion)**.** Assume that $\mu \in \mathcal{A}$ and $\sigma \in \mathcal{S}$. Let W be a real-valued Wiener martingale.

1. Solve
$$dS_t = S_t\left[\mu_t\,dt + \sigma_t\,dW_t\right], \quad S(0) = s. \tag{1.10}$$
Hint: Solve this first in the case that μ and σ are real constants. Apply Itô's formula to the process S and the function $x \mapsto \ln x$.

2. Is the function $t \mapsto S_t$ continuous? Why?

3. Calculate $d(1/S_t)$, assuming $s \neq 0$.

4. With B given by (1.9) calculate $d(S_t/B_t)$.

Exercise 1.40 (Multi-dimensional gBm)**.** Let W be an $\mathbb{R}^d$-valued Wiener martingale. Let $\mu \in \mathcal{A}^m$ and $\sigma \in \mathcal{S}^{m\times d}$. Consider the stochastic processes $S_i = (S_i(t))_{t\in[0,T]}$ given by
$$dS^i_t = S^i_t\mu^i_t\,dt + S^i_t\sum_{j=1}^m \sigma^{ij}_t\,dW^j_t,\ S^i_0 = s_i,\ i = 1,\ldots,m. \tag{1.11}$$

1. Solve (1.11) for $i = 1,\ldots,m$.
Hint: Proceed as when solving (1.10). Start by assuming that μ and σ are constants. Apply the multi-dimensional Itô formula to the process S_i and the function $x \mapsto \ln(x)$. Note that the process S_i is just $\mathbb{R}$-valued so the multi-dimensionality only comes from W being $\mathbb{R}^d$ valued.

2. Is the function $t \mapsto S^i_t$ continuous? Why?

Exercise 1.41 (Ornstein–Uhlenbeck process)**.** Let $a, b, \sigma \in \mathbb{R}$ be constants such that $b > 0, \sigma > 0$. Let W be a real-valued Wiener martingale.

1. Solve
$$dr_t = (b - ar_t)\,dt + \sigma_t\,dW_t, \quad r(0) = r_0. \tag{1.12}$$
Hint: Apply Itô's formula to the process r and the function $(t,x) \mapsto e^{at}x$.

2. Is the function $t \mapsto r_t$ continuous? Why?

3. Calculate $\mathbb{E}[r_t]$ and $\mathbb{E}[r_t^2]$.

4. What is the distribution of r_t?

Exercise 1.42. If X is a Gaussian random variable with $\mathbb{E}[X] = \mu$ and $\mathrm{Var}(X) = \mathbb{E}[X^2 - (\mathbb{E}[X])^2] = \sigma^2$ then we write $X \sim N(\mu, \sigma^2)$.

Fact. If $X \sim N(0, \sigma_X^2)$, $Y \sim N(0, \sigma_Y^2)$ are independent then $X + Y \sim N(0, \sigma_X^2 + \sigma_Y^2)$.

Show that if $X \sim N(\mu, \sigma^2)$ then $\mathbb{E}[e^X] = e^{\mu + \frac{\sigma^2}{2}}$.

Exercise 1.43. Consider the SDE
$$dX^{t,x}_s = b(X^{t,x}_s)\,ds + \sigma(X^{t,x}_s)\,dW_s,\ \ t \leq s \leq T,\ \ X^{t,x}_t = x\,.$$
Assume it has a unique strong solution i.e. if $Y^{t,x}_s$ is another process that satisfies the SDE then
$$\mathbb{P}\left[\sup_{t\leq s\leq T}|X^{t,x}_s - Y^{t,x}_s| > 0\right] = 0\,.$$
Show that then the *flow property* holds i.e. for $0 \leq t \leq t' \leq T$ we have
$$X^{t,x}_s = X^{t',X^{t,x}_{t'}}_s, \qquad s \in [t', T].$$

1.5 Solutions to Exercises

Solution (Solution to Exercise 1.35)**.** Let

$$z(t) = \left(e^{-\int_0^t \lambda(r)dr}\right) \int_0^t \lambda(s)y(s)\,ds.$$

Then, almost everywhere in I,

$$z'(t) = \lambda(t)e^{-\int_0^t \lambda(r)dr} \underbrace{\left(y(t) - \int_0^t \lambda(s)y(s)\,ds\right)}_{\leq b(t)-a(t)},$$

by the inequality in our hypothesis. Hence for a.a. $s \in I$

$$z'(s) \leq \lambda(s)e^{-\int_0^s \lambda(r)dr} \left(b(s) - a(s)\right).$$

Integrating from 0 to t and using the fundamental theorem of calculus (which gives us $\int_0^t z'(s)\,ds = z(t) - z(0) = z(t)$) we obtain

$$\begin{aligned}\int_0^t \lambda(s)y(s)\,ds &\leq e^{\int_0^t \lambda(r)dr} \int_0^t \lambda(s)e^{-\int_0^s \lambda(r)dr} \left(b(s) - a(s)\right)\,ds \\ &= \int_0^t \lambda(t)e^{\int_s^t \lambda(r)dr} \left(b(s) - a(s)\right)\,ds.\end{aligned}$$

Using the left hand side of above inequality as the right hand side in the inequality in our hypothesis we get

$$y(t) + a(t) \leq b(t) + \int_0^t \lambda(s)e^{\int_s^t \lambda(r)dr} \left(b(s) - a(s)\right)\,ds,$$

which is the first conclusion of the lemma. Assume now further that b is monotone increasing and a nonnegative. Then

$$\begin{aligned}y(t) + a(t) &\leq b(t) + b(t) \int_0^t \lambda(s)e^{\int_s^t \lambda(r)dr}\,ds \\ &= b(t) + b(t) \int_0^t -de^{\int_s^t \lambda(r)dr} = b(t) + b(t)\left(-1 + e^{\int_0^t \lambda(r)dr}\right) \\ &= b(t)e^{\int_0^t \lambda(r)\,dr}.\end{aligned}$$

Solution (Solution to Exercise 1.36)**.** Let $n \in \mathbb{N}$.

1. The sequence $b_n := \inf\{a_k : k \geq n\}$ is monotone increasing as $\{a_k : k \geq n+1\}$ is a subset of $\{a_k : k \geq n\}$, hence $b_n \leq b_{n+1}$. Additionally, the sequence is also bounded by the same bounds as the initial sequence (a_n). A monotone and bounded sequence of real numbers must converge and hence we can conclude that $\liminf_{n\to\infty} a_n$ exists.

2. It follows from the definition of infimum that there exists a sequence $i = i(n) \geq n$ such that

$$b_n = \inf\{a_k : k \geq n\} \leq a_i < \inf\{a_k : k \geq n\} + \frac{1}{n} = b_n + \frac{1}{n}.$$

   The sequence of indices $\big(i(n)\big)_{n\in\mathbb{N}}$ might not be monotone, but since $i(n) \geq n$ it is always possible to select its subsequence, say $\big(j(n)\big)_{n\in\mathbb{N}}$, that is monotone.

   Since $|a_{i(n)} - b_n| \to 0$ and $(b_n)_{n\in\mathbb{N}}$ converges to $\liminf_{n\to\infty} a_n$, then so does $(a_{i(n)})_n$. As $(a_{j(n)})_n$ is a subsequence of $(a_{i(n)})_n$ the same is true for $(a_{j(n)})_n$. Hence the claim follows.

Solution (Solution to Exercise 1.37)**.** We will show the result when $b > 0$, assuming that the $\sup$ takes a finite value. Let $f^* := \sup_{x \in X} f(x)$, and $V^* := \sup_{x \in X} \{a + bf(x)\}$.

To show that $V^* = a + bf^*$, we start by showing that $V^* \leq a + bf^*$.

Note that for all $x \in X$ we have $a + bf^* \geq a + bf(x)$, that is, $a + bf^*$ is an upper bound for the set $\{y : y = a + bf(x) \text{ for some } x \in X\}$. As a consequence, its least upper bound V^* must be such that $a + bf^* \geq V^* = sup_{x \in X}\{a + bf(x)\}$.

To show the converse, note that from the definition of f^* as a supremum (see Definition 1.1), we have that for any $\varepsilon > 0$ there must exist a $\overline{x}^\varepsilon \in X$ such that $f(\overline{x}^\varepsilon) > f^* - \varepsilon$.

Hence $a + bf(\overline{x}^\varepsilon) > a + bf^* - b\varepsilon$. Since $\overline{x}^\varepsilon \in X$, it is obvious that $V^* \geq a + bf(\overline{x}^\varepsilon)$. Hence $V^* \geq a + bf^* - b\varepsilon$. Since ε was arbitrarily chosen, we have our result: $V^* \geq a + bf^*$.

Solution (Solution to Exercise 1.38)**.** Let $t \in [0, \infty)$.

1. We are looking to solve:
$$B(t) = 1 + \int_0^t r(s)\, ds,$$
which is equivalent to
$$\frac{dB(t)}{dt} = r(t)B(t) \text{ for almost all } t,\ B(0) = 1.$$
Let us calculate (using chain rule and the above equation)
$$\frac{d}{dt}\left[\ln B(t)\right] = \frac{dB(t)}{dt} \cdot \frac{1}{B(t)} = r(t).$$
Integrating both sides and using the fundamental theorem of calculus
$$\ln B(t) - \ln B(0) = \int_0^t r(s)\, ds$$
and hence
$$B(t) = \exp\left(\int_0^t r(s)\, ds\right).$$

2. First we note that for any function f integrable on $[0, \infty)$ we have that the map $t \mapsto \int_0^t f(x)\, dx$ is absolutely continuous in t and hence it is continuous. The function $x \mapsto e^x$ is continuous and composition of continuous functions is continuous. Hence $t \mapsto B(t)$ must be continuous.

3. There are many ways to do this. We can start with (1.9) and use chain rule:
$$\frac{d}{dt}\left[\frac{1}{B(t)}\right] = \frac{dB(t)}{dt} \cdot \left(-\frac{1}{B^2(t)}\right) = -r(t)\left(-\frac{1}{B(t)}\right)$$
and so
$$d\left(\frac{1}{B(t)}\right) = -r(t)\frac{1}{B(t)}dt.$$
Or we can start with the solution that we have calculated write
$$\begin{aligned}\frac{d}{dt}\left[\frac{1}{B(t)}\right] &= \frac{d}{dt}\exp\left(-\int_0^t r(s)ds\right)\\ &= -r(t)\exp\left(-\int_0^t r(s)ds\right) = -r(t)\left(-\frac{1}{B(t)}\right)\end{aligned}$$
which leads to the same conclusion again.

Solution (Solution to Exercise 1.39). 1. We follow the hint (but skip directly to the general μ and σ). From Itô's formula:

$$d(\ln S(t) = \frac{1}{S(t)}dS(t) - \frac{1}{2}\frac{1}{S^2(t)}dS(t)\cdot dS(t) = \left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \mu(t)dW(t).$$

Now we write this in the full integral notation:

$$\ln S(t) = \ln S(0) + \int_0^t \left[\mu(s) - \frac{1}{2}\sigma^2(s)\right]ds + \int_0^t \mu(s)dW(s).$$

Hence

$$S(t) = s\exp\left(\int_0^t \left[\mu(s) - \frac{1}{2}\sigma^2(s)\right]ds + \int_0^t \mu(s)dW(s)\right). \tag{1.13}$$

Now this is the correct result but using invalid application of Itô's formula. If we want a full proof we call (1.13) a guess and we will now check that it satisfies (1.10). To that end we apply Itô's formula to $x \mapsto s\exp(x)$ and the Itô process

$$X(t) = \int_0^t \left[\mu(s) - \frac{1}{2}\sigma^2(s)\right]ds + \int_0^t \mu(s)dW(s).$$

Thus

$$\begin{aligned} dS(t) = d(f(X(t)) &= se^{X(t)}dX(t) + \frac{1}{2}se^{X(t)}dX(t)dX(t) \\ &= S(t)\left[\left(\mu(t) - \frac{1}{2}\sigma^2(t)\right)dt + \mu(t)dW(t)\right] + \frac{1}{2}S(t)\sigma^2(t)dt. \end{aligned}$$

Hence we see that the process given by (1.13) satisfies (1.10).

2. The continuity question is now more intricate than in the previous exercise due to the presence of the stochastic integral. From stochastic analysis in finance you know that Z given by

$$Z(t) := \int_0^t \sigma(s)dW(s)$$

is a continuous stochastic process. Thus there is a set $\Omega' \in \mathcal{F}$ such that $\mathbb{P}(\Omega') = 1$ and for each $\omega \in \Omega'$ the function $t \mapsto S(\omega, t)$ is continuous since it's a composition of continuous functions.

3. If $s \neq 0$ then $S(t) \neq 0$ for all t. We can thus use Itô's formula

$$\begin{aligned} d\left(\frac{1}{S(t)}\right) &= -\frac{1}{S^2(t)}dS(t) + \frac{1}{S^3(t)}dS(t)dS(t) \\ &= -\frac{1}{S(t)}\left[\mu(t)dt + \sigma(t)dW(t)\right] + \frac{1}{S(t)}\sigma^2(t)dt \\ &= \frac{1}{S(t)}\left[\left(-\mu(t) + \sigma^2(t)\right)dt - \sigma(t)dW(t)\right]. \end{aligned}$$

4. We calculate this with Itô's product rule:

$$\begin{aligned} d\left(\frac{S(t)}{B(t)}\right) &= S(t)d\left(\frac{1}{B(t)}\right) + \frac{1}{B(t)}dS(t) + dS(t)d\left(\frac{1}{B(t)}\right) \\ &= -r(t)\frac{S(t)}{B(t)}dt + \mu(t)\frac{S(t)}{B(t)}dt + \sigma(t)\frac{S(t)}{B(t)}dW(t) \\ &= \frac{S(t)}{B(t)}\left[\left(\mu(t) - r(t)\right)dt + \sigma(t)dW(t)\right]. \end{aligned}$$

Solution (Solution to Exercise 1.40).

1. We Itô's formula to the function $x \mapsto \ln(x)$ and the process S_i. We thus obtain, for $X_i(t) := \ln(S_i(t))$, that

$$\begin{aligned} dX_i(t) = d\ln(S_i(t)) &= \frac{1}{S_i(t)} dS_i(t) - \frac{1}{2} \frac{1}{S_i^2(t)} dS_i(t) dS_i(t) \\ &= \mu_i(t)dt + \sum_{j=1}^n \sigma_{ij}(t) dW_j(t) - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2(t) dt \\ &= \left[\mu_i(t) - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2(t) \right] dt + \sum_{j=1}^n \sigma_{ij}(t) dW_j(t). \end{aligned}$$

Hence

$$\begin{aligned} X_i(t) - X_i(0) &= \ln S_i(t) - \ln S_i(t) \\ &= \int_0^t \left[\mu_i(s) - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2(s) \right] ds + \sum_{j=1}^n \int_0^t \sigma_{ij}(s) dW_j(s). \end{aligned}$$

And so

$$S_i(t) = S_i(0) \exp \left\{ \int_0^t \left[\mu_i(s) - \frac{1}{2} \sum_{j=1}^n \sigma_{ij}^2(s) \right] ds + \sum_{j=1}^n \int_0^t \sigma_{ij}(s) \, dW_j(s) \right\}.$$

2. Using the same argument as before and in particular noticing that for each j the function $t \mapsto \int_0^t \sigma_{ij}(s) dW_j(s)$ is continuous for almost all $\omega \in \Omega$ we get that $t \mapsto S_i(t)$ is almost surely continuous.

2 Introduction to Stochastic Control

2.1 A motivating example from Merton's problem

In this part we give a motivating example to introduce the problem of dynamic asset allocation and stochastic optimization. We will not be particularly rigorous in these calculations.

The market Consider an investor can invest in a two asset Black-Scholes market: a risk-free asset ("bank" or "Bond") with rate of return $r > 0$ and a risky asset ("stock") with mean rate of return $\mu > r$ and constant volatility $\sigma > 0$. Suppose that the price of the risk-free asset at time t, B_t, satisfies

$$\frac{dB_t}{B_t} = r\,dt \quad \text{or} \quad B_t = B_0 e^{rt}, \qquad t \geq 0.$$

The price of the stock evolves according to the following SDE:

$$\frac{dS_t}{S_t} = \mu\,dt + \sigma\,dW_t,$$

where $(W_t)_{t\geq 0}$ is a standard one-dimensional Brownian motion one the filtered probability space $(\Omega, \mathcal{F}, \mathbb{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$.

The agent's wealth process and investments Let X_t^0 denote the investor's wealth in the bank at time $t \geq 0$. Let π_t denote the wealth in the risky asset. Let $X_t = X_t^0 + \pi_t$ be the investor's total wealth. The investor has some initial capital $X_0 = x > 0$ to invest. Moreover, we also assume that the investor saves / consumes wealth at rate C_t at time $t \geq 0$.

There are three popular possibilities to describe the investment in the risky asset:

(i) Let ξ_t denote the number of units stocks held at time t (allow to be fractional and negative),

(ii) the value in units of currency $\pi_t = \xi_t S_t$ invested in the risky asset at time t,

(iii) the fraction $\nu_t = \frac{\pi_t}{X_t}$ of current wealth invested in the risky asset at time t.

The investment in the bond is then determined by the accounting identity $X_t^0 = X_t - \pi_t$. The parametrizations are equivalent as long as we consider *only* positive wealth processes (which we shall do). The gains/losses from the investment in the stock are then given by

$$\xi_t\,dS_t, \qquad \frac{\pi_t}{S_t}\,dS_t, \qquad \frac{X_t\nu_t}{S_t}\,dS_t\,.$$

The last two ways to describe the investment are especially convenient when the model for S is of the exponential type, as is the Black-Scholes one. Using (ii),

$$\begin{aligned} X_t &= x + \int_0^t \frac{\pi_s}{S_s}\,dS_s + \int_0^t \big(X_s - \pi_s\big)\frac{dB_s}{B_s} - \int_0^t C_s\,ds \\ &= x + \int_0^t \big[\pi_s(\mu - r) + rX_s - C_s\big]\,ds + \int_0^t \pi_s\sigma\,dW_s \end{aligned}$$

or in differential form

$$dX_t = \big[\pi_t(\mu - r) + rX_t - C_t\big]\,dt + \pi_t\sigma\,dW_t, \qquad X_0 = x\,.$$

Alternatively, using (iii), the equation simplifies even further.[2] Recall $\pi = \nu X$.

$$\begin{aligned}dX_t &= X_t\nu_t\frac{dS_t}{S_t} + X_t\big(1-\nu_t\big)\frac{dB_t}{B_t} - C_t\,dt\\ &= \big[X_t\big(\nu_t(\mu - r) + r\big) - C_t\big]\,dt + X_t\nu_t\sigma\,dW_t.\end{aligned}$$

We can make a further simplification and obtain an SDE in "geometric Brownian motion" format if we assume that the consumption C_t can be written as a fraction of the total wealth, i.e. $C_t = \kappa_t X_t$. Then

$$dX_t = X_t\big[\nu_t(\mu - r) + r - \kappa_t\big]\,dt + X_t\nu_t\sigma\,dW_t\,. \tag{2.1}$$

Exercise 2.1. Assuming that all coefficients in SDE (2.1) are integrable, solve the SDE for X and hence show $X > 0$ when $X_0 = x > 0$.

The optimization problem The investment allocation/consumption problem is to choose the best investment possible in the stock, bond and at the same time consume the wealth optimally. How to translate the words "best investment" into a mathematical criteria?

Classical modeling for describing the behavior and preferences of agents and investors are: *expected utility* criterion and *mean-variance* criterion.

In the *first criterion* relying on the theory of choice in uncertainty, the agent compares random incomes for which he knows the probability distributions. Under some conditions on the preferences, Von Neumann and Morgenstern show that they can be represented through the expectation of some function, called *utility*. Denoting it by U, the utility function of the agent, the random income X is preferred to a random income X' if $\mathbb{E}[U(X)] \geq \mathbb{E}[U(X')]$. The deterministic utility function U is nondecreasing and concave, this last feature formulating the risk aversion of the agent.

Example 2.2 (Examples of utility functions)**.** The most common utility functions are

- Exponential utility: $U(x) = -e^{-\alpha x}$, the parameter $\alpha > 0$ is the risk aversion.
- Log utility: $U(x) = \ln(x)$
- Power utility: $U(x) = (x^\gamma - 1)/\gamma$ for $\gamma \in (-\infty, 0) \cup (0, 1)$.
- Iso-elastic: $U(x) = x^{1-\rho}/(1-\rho)$ for $\rho \in (-\infty, 0) \cup (0, 1)$.

In this portfolio allocation context, the criterion consists of maximizing the expected utility from consumption and from terminal wealth. In the **the finite time-horizon case**: $T < \infty$, this is

$$\sup_{\nu, C}\mathbb{E}\left[\int_0^T U\big(C_t\big)\,dt + U\big(X_t^{\nu,C}\big)\right]\ , \ \text{where (2.1) gives } X_t^{\nu,C} = X_t. \tag{2.2}$$

[2]Note that, if ν_t expresses the fraction of the total wealth X invested in the stock, then the fraction of wealth invested in the bank account is simply $1 - \nu_t$.

Without consumption, i.e. $\forall\, t$ we have $C(t) = 0$, the optimization problem could be written as

$$\sup_{\nu} \mathbb{E}\left[U\big(X_t^{\nu}\big)\right]\ , \ \text{ where (2.1) gives } X_t^{\nu} = X_t. \tag{2.3}$$

Note that the maximization is done under the expectation.

In the **infinite time-horizon case**: $T = \infty$. In our context the optimization problem is written as (recall that $C_t = \kappa_t X_t^{\nu,\kappa}$)

$$\sup_{\kappa,\nu} \mathbb{E}\left[\int_0^{\infty} e^{-\gamma t} U\big(\kappa_t X_t^{\nu,\kappa}\big)\, dt\, , \ \text{ with (2.1) giving } X_t = X_t^{\nu,\kappa}.\right] \tag{2.4}$$

Let us go back to the **finite horizon case**: $T < \infty$. The *second criterion* for describing the behavior and preferences of agents and investors, the mean-variance criterion, relies on the assumption that the preferences of the agent depend only on the expectation and variance of his random incomes. To formulate the feature that the agent likes wealth and is risk-averse, the mean-variance criterion focuses on mean-variance-efficient portfolios, i.e. minimizing the variance given an expectation.

In our context and assuming that there is no consumption, i.e. $\forall\, t$ we have $C_t = 0$, then the optimization problem is written as

$$\inf_{\nu} \left\{\mathrm{Var}\big(X_T^{\nu}\big)\ :\ \mathbb{E}[X_T^{\nu}] = m, \quad m \in (0,\infty)\right\}.$$

We shall see that this problem may be reduced to the resolution of a problem in the form (2.2) for the quadratic utility function: $U(x) = \lambda - x^2$, $\lambda \in \mathbb{R}$.

2.1.1 Basic elements of a stochastic control problem

The above investment-consumption problem and its variants (is the so-called "Merton problem" and) is an example of a stochastic optimal control problem. Several key elements, which are common to many stochastic control problems, can be seen.

These include:

Time horizon. The time horizon in the investment-consumption problem may be finite or infinite, in the latter case we take the time index to be $t \in [0,\infty)$. We will also consider problems with finite horizon: $[0,T]$ for $T \in (0,\infty)$; and indefinite horizon: $[0,\tau]$ for some stopping time τ (for example, the first exit time from a certain set).

(Controlled) State process. The state process is a stochastic process which describes the state of the physical system of interest. The state process is often given by the solution of an SDE, and if the control process appears in the SDE's coefficients it is called a *controlled stochastic differential equation*. The evolution of the state process is influenced by a control. The state process takes values in a set called the state space, which is typically a subset of $\mathbb{R}^d$. In the investment-consumption problem, the state process is the wealth process $X^{\nu,C}$ in (2.1).

Control process. The control process is a stochastic process, chosen by the "controller" to influence the state of the system. For example, the controls in the investment-consumption problem are the processes $(\nu_t)_t$ and $(C_t)_t$ (see (2.1)).

We collect all the control parameters into one process denoted $\alpha = (\nu, C)$. The control process $(\alpha_t)_{t\in[0,T]}$ takes values in an action set A. The action set can be a complete separable metric space but most commonly $A \in \mathcal{B}(\mathbb{R}^m)$.

For the control problem to be meaningful, it is clear that the choice of control must allow for the state process to exist and be determined uniquely. More generally, the control may be forced satisfy further constraints like "no short-selling" (i.e. $\pi(t) \geq 0$) and or the control space varies with time. In the financial context, the control map at time t should be decided at time t based on the available information $\mathcal{F}_t$. This translates into requiring the control process to be adapted.

Admissible controls. Typically, only controls which satisfy certain "admissibility" conditions can be considered by the controller. These conditions can be both technical, for example, integrability or smoothness requirements, and physical, for example, constraints on the values of the state process or controls. For example, in the investment-consumption problem we will only consider processes $X^{\nu,C}$ for which a solution to (2.1) exists. We will also require $C_t \geq 0$ and that the investor have non-negative wealth at all times, which places further restrictions on the class of allowable controls.

Objective function. There is some cost/gain associated with the system, which may depend on the system state itself and on the control used. The objective function contains this information and is typically expressed as a function $J(x, \alpha)$ (or in finite-time horizon case $J(t, x, \alpha)$), representing the expected total cost/gain starting from system state x (at time t in finite-time horizon case) if control process α is implemented.

For example, in the setup of (2.3) the *objective functional* (or gain/cost map) is

$$J(0, x, \nu) = \mathbb{E}\left[U\big(X^\nu(T)\big)\right], \tag{2.5}$$

as it denotes the reward associated with initial wealth x and portfolio process ν. Note that in the case of no-consumption, and given the remaining parameters of the problem (i.e. μ and σ), both x and ν determine by themselves the value of the reward.

Value function. The value function describes the value of the maximum possible gain of the system (or minimal possible loss). It is usually denoted by v and is obtained, for initial state x (or (t, x) in finite-time horizon case), by optimizing the cost over all admissible controls. The goal of a stochastic control problem is to find the value function v and find a control α^* whose cost/gain attains the minimum/maximum value: $V(x) = J(x, \alpha^*)$ for starting state x. For completeness sake, from (2.3) and (2.5), if ν^* is the optimal control, then we have the *value function*

$$V(x) = \sup_\nu \mathbb{E}\left[U\big(X^\nu(T)\big)\right] = \sup_\nu J(x, \nu) = J(x, \nu^*). \tag{2.6}$$

Typical questions of interest Typical questions of interest in Stochastic control problems include:

- Is there an optimal control?
- Is there an optimal Markov control?
- How can we find an optimal control?
- How does the value function behave?
- Can we compute or approximate an optimal control numerically?

There are of course many more and, before we start, we need to review some concepts of stochastic analysis that will help in the rigorous discussion of the material in this section so far.

2.2 Controlled diffusions

We now introduce controlled SDEs with a finite time horizon $T > 0$; the infinite-horizon case is discussed later. Again, $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space with filtration $(\mathcal{F}_t)$ and a d'-dimensional Wiener process W compatible with this filtration.

We are given an action set A (in general separable complete metric space) and let $\mathcal{U}_0$ be the set of all A-valued progressively measurable processes, the controls. The controlled state is defined through an SDE as follows. Let

$$b : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^d \quad \text{and} \quad \sigma : [0,T] \times \mathbb{R}^d \times A \to \mathbb{R}^{d\times d'}$$

be measurable functions satisfying the Lipschitz and linear growth conditions[3].

Assumption 2.3. There is a constant K and an integrable process $\kappa = \kappa_t$ such that for any t, x, y, a we have

$$|b(t,x,a) - b(t,y,a)| + |\sigma(t,x,a) - \sigma(t,y,a)| \le K|x-y|, \tag{2.7}$$

$$|b(t,x,a)| + |\sigma(t,x,a)| \le \kappa_t(1+|x|) \tag{2.8}$$

and $\mathbb{E}\int_0^t \kappa_s^2\, ds < \infty$ for any t.

Let $\mathcal{U} \subseteq \mathcal{U}_0$ be the subset of control processes for which we have Assumption 2.3. We will refer to this set as *admissible controls*. Note that in most of our examples $\alpha \in \mathcal{U}$ if and only if $\mathbb{E}\int_0^T \alpha_s^2\, ds < \infty$.

Given a fixed control $\alpha \in \mathcal{U}$, we consider the SDE for $0 \le t \le T \le \infty$ for $s \in [t,T]$

$$dX_s = b\big(s, X_s, \alpha_s\big)\, ds + \sigma\big(s, X_s, \alpha_s\big)\, dW_s, \quad X_t = \xi. \tag{2.9}$$

With Assumption 2.3 the SDE (2.9) is a special case of an SDE with random coefficients, see (1.5). As discussed in Section 1.3 and the results there, we have the following result.

Proposition 2.4 (Existence and uniqueness). *Let $t \in [0,T]$, $\xi \in L^2(\mathcal{F}_t)$ and $\alpha \in \mathcal{U}_0$. Then SDE (2.9) has a unique (strong) Markov solution $X = X^\alpha_{t,\xi}$ on the interval $[t,T]$ such that*

$$\sup_{\alpha\in\mathcal{U}} \mathbb{E} \sup_{s\in[t,T]} |X_s|^2 \le c(1+\mathbb{E}|\xi|^2)\,.$$

Moreover, the solution has the properties listed in Proposition 1.28.

2.3 Formulation of stochastic control problems

In this section we revisit the ideas of the opening one and give a stronger mathematical meaning to the general setup for optimal control problems. We distinguish the finite time horizon $T < \infty$ and the infinite time horizon $T = \infty$, the functional to optimize must differ.

[3] Note that the Lipschitz condition for a certain variable implies that the linear growth condition is satisfied for that variable. One is not able to conclude anything about the (possibly) other variables. From (2.7) one cannot conclude (2.8) as the latter assumes linear growth in the a variable. In mathematical terms, from (2.7) on can only conclude that $|b(t,x,a)| + |\sigma(t,x,a)| \le K_{t,a}(1+|x|)$ with the associated constant K depending on t and a.

In general, texts either discuss maximization or a minimization problems. Using analysis results, it is easy to jump between minimization and maximization problems: $\max_x f(x) = -\min_x -f(x)$ and the x^* that maximizes f is the same one that minimizes $-f$ (draw a picture to convince yourself).

Finite time horizon

Let

$$J(t,\xi,\alpha) := \mathbb{E}\left[\int_t^T f(s, X_s^{\alpha,t,\xi}, \alpha_s)\, ds + g(X_T^{\alpha,t,\xi})\right],$$

where $X_{t,\xi}$ solves (2.9) (with initial condition $X(t) = \xi$). The J here is called the *objective functional*. We refer to f as the *running gain* (or, if minimizing, *running cost*) and to g as the *terminal gain* (or *terminal cost*).

We will ensure the good behavior of J through the following assumption.

Assumption 2.5. There is $K > 0$ such that for all t, x, y, a we have

$$|g(x) - g(y)| + |f(t,x,a) - f(t,y,a)| \le K|x-y|,$$

$$|f(t,0,a)| \le K\,.$$

Note that this assumption is too restrictive for many of the problems we want to consider. Indeed a typical $g(x) = x^2$ is not covered by such assumption. However it makes the proofs much simpler. For bigger generality consult e.g. [Kry80].

Exercise 2.6 (The objective functional J is well-defined)**.** Let Assumptions 2.3 and 2.5 hold. Show that there is $c_T > 0$ such that $|J(\cdot,\cdot,\alpha)| < c_T(1+|x|)$ for any $\alpha \in \mathcal{U}[t,T]$.

The optimal control problem formulation We will focus on the following stochastic control problem. Let $t \in [0,T]$ and $x \in \mathbb{R}^d$. Let

$$(P)\begin{cases} v(t,x) := \sup\limits_{\alpha\in\mathcal{U}[t,T]} J(t,x,\alpha) = \sup\limits_{\alpha\in\mathcal{U}[t,T]} \mathbb{E}\left[\displaystyle\int_t^T f\big(s, X_s^{\alpha,t,x}, \alpha_s\big)\, ds + g\big(X_T^{\alpha,t,x}\big)\right] \\ \text{and } X^{\alpha,t,x} \text{ solves (2.9) with } X_t^{\alpha,t,x} = x. \end{cases}$$

The solution to the problem (P), is the *value function*, denoted by v. One of the mathematical difficulties in stochastic control theory is that we don't even know at this point whether v is measurable or not.

In many cases there is no optimal control process α^* for which we would have $v(t,x) = J(t,x,\alpha^*)$. Recall that v is the value function of the problem (P). However there is always an ε-optimal control (simply by definition of supremum).

Definition 2.7 (ε-optimal controls)**.** Take $t \in [0,T]$ and $x \in \mathbb{R}^m$. Let $\varepsilon \ge 0$. A control $\alpha^\varepsilon \in \mathcal{U}_{ad}[t,T]$ is said to be ε*-optimal* if

$$v(t,x) \le \varepsilon + J(t,x,\alpha^\varepsilon)\,. \tag{2.10}$$

Lemma 2.8 (Lipschitz continuity in x of the value function)**.** *If Assumptions 2.3 and 2.5 hold then there exists $C_T > 0$ such that for all $t \in [0,T]$ and $x, y \in \mathbb{R}^d$ we have*

$$|v(t,x) - v(t,y)| \le C_T|x-y|\,.$$

Proof. The first step is to show that there is $C_T > 0$ such that for any $\alpha \in \mathcal{U}$ we have

$$I := |J(t,x,\alpha) - J(t,y,\alpha)| \le C_T|x-y|\,.$$

We note that due to Hölder's and Young's inequalities

$$\begin{aligned}I^2 &\le \mathbb{E}\left[2\left(\int_t^T f(s,X_s^{t,x,\alpha},\alpha_s) - f(s,X_s^{t,y,\alpha},\alpha_s)\,ds\right)^2 + 2\big(g(X_T^{t,x,\alpha}) - g(X_T^{t,y,\alpha})\big)^2\right]\\ &\le \mathbb{E}\left[2T\int_t^T |f(s,X_s^{t,x,\alpha},\alpha_s) - f(s,X_s^{t,y,\alpha},\alpha_s)|^2\,ds + 2\big|g(X_T^{t,x,\alpha}) - g(X_T^{t,y,\alpha})\big|^2\right].\end{aligned}$$

Using Assumption 2.5 (Lipschitz continuity in x of f and g) we get

$$I^2 \le 2TK^2\int_t^T \mathbb{E}|X_s^{t,x,\alpha} - X_s^{t,y,\alpha}|^2\,ds + 2K^2\mathbb{E}\big|X_s^{t,x,\alpha} - X_s^{t,y,\alpha}\big|^2\,.$$

Then, using Proposition 1.28, we get

$$I^2 \le 2(T^2+1)K^2 \sup_{t\le s\le T}\mathbb{E}|X_s^{t,x,\alpha} - X_s^{t,y,\alpha}|^2 \le C_T|x-y|^2\,.$$

We now need to apply this property of J to the value function v. Let $\varepsilon > 0$ be arbitrary and fixed. Then there is $\alpha^\varepsilon \in \mathcal{U}$ such that $v(t,x) \le \varepsilon + J(t,x,\alpha^\varepsilon)$. Moreover $v(t,y) \ge J(t,y,\alpha^\epsilon)$. Thus

$$v(t,x) - v(t,y) \le \varepsilon + J(t,x,\alpha^\varepsilon) - J(t,y,\alpha^\epsilon) \le \varepsilon + C_T|x-y|\,.$$

With $\varepsilon > 0$ still the same and fixed there would be $\beta^\varepsilon \in \mathcal{U}$ such that $v(t,y) \le \varepsilon + J(t,y,\beta^\varepsilon)$. Moreover $v(t,x) \ge J(t,x,\beta^\epsilon)$ and so

$$v(t,y) - v(t,x) \le \varepsilon + J(t,y,\beta^\varepsilon) - J(t,x,\beta^\epsilon) \le \varepsilon + C_T|x-y|\,.$$

Hence $-\varepsilon - C_T|x-y| \le v(t,x) - v(t,y) \le \varepsilon + C_T|x-y|$. Letting $\varepsilon \to 0$ concludes the proof. □

An important consequence of this is that if we fix t then $x \mapsto v(t,x)$ is measurable (as continuous functions are measurable).

2.4 Exercises

Exercise 2.9 (Further moment bounds)**.** Fix $k \ge 1$. Assume that for all there is κ_t such that $\mathbb{E}\left(\int_0^T \kappa_t^{\frac{2k}{k-1}}\,dt\right)^{k-1} < \infty$ and for any (t,x,a)

$$|b(t,x,a)| + |\sigma(t,x,a)| \le \kappa_t(1+|x|)\,.$$

Let X be a solution to (2.9) for some $\xi \in \mathbb{R}^d$. Show that there is $C = C_{k,\kappa,T}$ such that

$$\sup_{s\in[0,T]} \mathbb{E}\big[\,|X_s|^{2k}\big] < C_T(1+|\xi|^k)\,.$$

Exercise 2.10 (Bound on J)**.** Assume that the conditions of Exercise (2.9) hold. Show that, if there exist constants $M > 0$ and $k > 0$, such that for any $s \in [0,\ T]$, $x \in \mathbb{R}^d$ and $a \in A$,

$$|f(s,x,\nu)| + |g(x)| \le M(1+|x|^{2k})$$

then there exists a constant $C = C_{T,M,k,\kappa}$ such that $|J(t,x,\alpha)| < C(1+|x|^{2k})$ for any $\alpha \in \mathcal{U}$ and any $x \in \mathbb{R}^d$.

2.5 Solutions to Exercises

Solution (to Exercise 2.9)**.** Take absolute value in (2.9) and raise to power of $2k$. Then

$$|X_s|^{2k} \le c_k|\xi|^{2k} + c_k\bigg|\int_t^s b(t,X_r,\alpha_r)\,dr\bigg|^{2k} + c_k\bigg|\int_t^s \sigma(t,X_r,\alpha_r)\,dW_r\bigg|^{2k}.$$

Taking expectation we get

$$\mathbb{E}|X_s|^{2k} \le c_k|\xi|^{2k} + c_k\mathbb{E}\bigg|\int_t^s b(t,X_r,\alpha_r)\,dr\bigg|^{2k} + c_k\mathbb{E}\bigg|\int_t^s \sigma(t,X_r,\alpha_r)\,dW_r\bigg|^{2k} =: c_k|\xi|^{2k} + I_1 + I_2\,.$$

We start by applying Hölder's inequality to I_1:

$$I_1 = c_k\mathbb{E}\bigg|\int_t^s b(t,X_r,\alpha_r)\,dr\bigg|^{2k} \le C_{k,T}\mathbb{E}\bigg(\int_t^s |b(t,X_r,\alpha_r)|^2\,dr\bigg)^k$$

Then we need our growth assumption to see that

$$I_1 \le C_{k,T}\mathbb{E}\bigg(\int_t^s |b(t,X_r,\alpha_r)|^2\,dr\bigg)^k \le C_{k,T}\mathbb{E}\bigg(\int_t^s \kappa_r^2(1+|X_r|^2)\,dr\bigg)^k.$$

We now consider I_2: we use e.g. Burkholder–Davis–Gundy inequality to see

$$I_2 = c_k\mathbb{E}\bigg|\int_t^s \sigma(t,X_r,\alpha_r)\,dW_r\bigg|^{2k} \le C_k\mathbb{E}\bigg(\int_t^s |\sigma(t,X_r,\alpha_r)|^2\,dr\bigg)^k.$$

Now we apply our growth assumption to see that

$$I_2 \le C_k\mathbb{E}\bigg(\int_t^s \kappa_r^2(1+|X_r|^2)\,dr\bigg)^k.$$

With Hölder's inequality we see that

$$\begin{aligned}\mathbb{E}\bigg(\int_t^s \kappa_r^2(1+|X_r|^2)\,dr\bigg)^k &\le c_k\mathbb{E}\bigg(\int_t^s \kappa_r^{\frac{2k}{k-1}}\,dr\bigg)^{k-1}\mathbb{E}\bigg(\int_t^s (1+|X_r|^{2k})\,dr\bigg)\\ &\le C_{k,\kappa,T}\mathbb{E}\bigg(\int_t^s (1+|X_r|^{2k})\,dr\bigg).\end{aligned}$$

Altogether we now have

$$\mathbb{E}|X_s|^{2k} \le c_k|\xi|^{2k} + C_{k,\kappa,T}\int_t^s (1+\mathbb{E}|X_r|^{2k})\,dr\,.$$

Now take $y(s) := \mathbb{E}|X_{s+t}|^{2k}$ so that we have

$$y(s) \le C_{k,\kappa,T}(1+|\xi|^{2k}) + \int_0^s C_{k,\kappa,T}\,y(r)\,dr\,.$$

With Gronwall's inequality we see that $y(s) \le C_{k,\kappa,T}(1+|\xi|^{2k})e^{C_{k,\kappa,T}\,s}$ and this is exactly (after taking supremum)

$$\sup_{t\le s\le T}\mathbb{E}|X_s|^{2k} \le C(1+|\xi|^{2k})\,.$$

Solution (to Exercise 2.10)**.** We will write $X_s = X_s^{t,x,\alpha}$. Then

$$\begin{aligned}|J(t,x,\alpha)| &\le \mathbb{E}\left[\int_t^T |f(s,X_s,\alpha_s)|\,ds + |g(X_T)|\right] \le (1+T)\sup_{t\le s\le T}\mathbb{E}\left[|f(s,X_s,\alpha_s)| + |g(X_s)|\right]\\ &\le M(1+T)\sup_{t\le s\le T}\mathbb{E}\left[1+|X_s|^{2k}\right] \le C(1+|x|^{2k}),\end{aligned}$$

where, in the last step, we used the conclusion of Exercise 2.9.

3 Dynamic Programming and Hamilton-Jacobi-Bellman Equations

3.1 Dynamic Programming Principle

Dynamic programming (DP) is one of the most popular approaches to study the stochastic control problem (P). The main idea was originated from the so-called Bellman's principle, which states

> *An optimal policy has the property that whatever the initial state and initial decision are, the remaining decisions must constitute an optimal policy with regard to the state resulting from the first decision.*

The following is the statement of Bellman's principle / dynamic programming.

Theorem 3.1 (Bellman's principle / Dynamic programming). *For any $0 \leq t \leq \hat{t} \leq T$, for any $x \in \mathbb{R}^m$, we have*

$$v(t,x) = \sup_{\alpha \in \mathcal{U}[t,\hat{t}]} \mathbb{E}\left[\int_t^{\hat{t}} f\big(s, X_s^{\alpha,t,x}, \alpha_s\big)\, ds + v\big(\hat{t}, X_{\hat{t}}^{\alpha,t,x}\big)\Big| X_t^{\alpha,t,x} = x\right]. \tag{3.1}$$

The idea behind the dynamic programming principle is as follows. The expectation on the RHS of (3.1) represents the gain if we implement the time t until time $\hat{t}$ optimal strategy and then implement the time $\hat{t}$ until T optimal strategy. Clearly, this gain will be no larger than the gain associated with using the overall optimal strategy from the start (since we can apply the overall optimal control in both scenarios and obtain the LHS).

What equation (3.1) says is that if we determine the optimal strategy separately on each of the time intervals $[t,\hat{t}]$ and $[\hat{t},T]$ we get the same answer as when we consider the whole time interval $[t,T]$ at once. Underlying this statement, hides a deeper one: that if one puts the optimal stategy over $[t,\hat{t}\,)$ together with the optimal stategy over $[\hat{t},T]$ this is still an optimal strategy.

Note that without Lemma 2.8 we would not even be allowed to write (3.1) since we need $v(\hat{t}, X_{\hat{t}}^{\alpha,t,x})$ to be a random variable (so that we are allowed to take the expectation).

Let us now prove the Bellman principle.

Proof of Theorem 3.1. We will start by showing that $v(t,x) \leq$ RHS of (3.1). We note that with $\alpha \in \mathcal{U}[t,T]$ we have

$$J(t,x,\alpha) = \mathbb{E}\left[\int_t^{\hat{t}} f(s, X_s^\alpha, \alpha_s)ds + \int_{\hat{t}}^T f(s, X_s^\alpha, \alpha_s)ds + g(X_T^\alpha)\Big| X_t^\alpha = x\right].$$

We will use the tower property of conditional expectation and use the Markov property of the process. Let $\mathcal{F}^{X^\alpha}_{\hat t} := \sigma(X^\alpha_s : t \leq s \leq \hat t)$. Then

$$
\begin{aligned}
&J(t,x,\alpha)\\
&= \mathbb{E}\left[\int_t^{\hat t} f(s,X^\alpha_s,\alpha_s)ds + \mathbb{E}\left[\int_{\hat t}^T f(s,X^\alpha_s,\alpha_s)ds + g(X^\alpha_T)\Big|\mathcal{F}^{X^\alpha}_{\hat t}\right]\Big|X^\alpha_t = x\right]\\
&= \mathbb{E}\left[\int_t^{\hat t} f(s,X^\alpha_s,\alpha_s)ds + \mathbb{E}\left[\int_{\hat t}^T f(s,X^\alpha_s,\alpha_s)ds + g(X^\alpha_T)\Big|X^\alpha_{\hat t}\right]\Big|X^\alpha_t = x\right].
\end{aligned}
$$

Now, because of the flow property of SDEs,

$$
\mathbb{E}\left[\int_{\hat t}^T f(s,X^{\alpha,t,x}_s,\alpha_s)ds + g(X^{\alpha,t,x}_T)\Big|X^{\alpha,t,x}_{\hat t}\right] = J\left(\hat t, X^{\alpha,t,x}_{\hat t}, (\alpha_s)_{s\in[\hat t,T]}\right) \leq v\left(\hat t, X^{\alpha,t,x}_{\hat t}\right).
$$

Hence

$$
J(t,x,\alpha) \leq \sup_{\alpha\in\mathcal{U}} \mathbb{E}\left[\int_t^{\hat t} f(s,X^\alpha_s,\alpha_s)ds + v\left(\hat t, X^{\alpha,t,x}_{\hat t}\right)\Big|X^\alpha_t = x\right].
$$

Taking supremum over control processes α on the left shows that $v(t,x) \leq$ RHS of (3.1).

We now need to show that RHS of (3.1) $\leq v(t,x)$. Fix $\varepsilon > 0$. Then there is $\alpha^\varepsilon \in \mathcal{U}[t,\hat t]$ such that

$$
\text{RHS of (3.1)} \leq \varepsilon + \mathbb{E}\left[\int_t^{\hat t} f\big(s, X^{\alpha^\varepsilon,t,x}_s, \alpha^\varepsilon_s\big)\, ds + v\big(\hat t, X^{\alpha^\varepsilon,t,x}_{\hat t}\big)\Big|X^{\alpha^\varepsilon,t,x}_s = x\right].
$$

We now have to be careful so that we can construct an ε-optimal control which is progressively measurable on the whole $[t,T]$. To that end consider $\delta > 0$ (which we will fix in a moment) and take a collection of disjoint cubes $Q_i \subset \mathbb{R}^d$, each with a centre $x_i \in \mathbb{R}^d$, such that $Q_i \subset B_\delta(x_i)$ and such that $\bigcup_i Q_i = \mathbb{R}^d$. Then for each x_i there is $\alpha^{\varepsilon,i} \in \mathcal{U}(\hat t, T]$ such that $v(\hat t, x_i) \leq \varepsilon + J(\hat t, x_i, \alpha^{\varepsilon,i})$.

Let us write $X_s := X^{\alpha^\varepsilon,t,x}_s$ for brevity. If $X_{\hat t} \in Q_i$ then $|x_i - X_{\hat t}| < \delta$ and due to Lemma 2.8 we have

$$
|v(\hat t, X_{\hat t}) - v(\hat t, x_i)| \leq C_T|X_{\hat t} - x_i| < C_T\delta\,,
$$

where $C_T > 0$ is a constant of Lipschitz continuity which is independent of i and of ε. Furthermore (check the proof of Lemma 2.8)

$$
|J(\hat t, x_i, \alpha^{\varepsilon,i}) - J(\hat t, X_{\hat t}, \alpha^{\varepsilon,i})| \leq C_T\delta\,.
$$

Hence we get

$$
v(\hat t, X_{\hat t}) \leq v(\hat t, x_i) + C_t\delta \leq \varepsilon + J(\hat t, x_i, \alpha^{\varepsilon,i}) + C_T\delta \leq \varepsilon + J(\hat t, X_{\hat t}, \alpha^{\varepsilon,i}) + 2C_T\delta\,.
$$

We now fix δ so that $2C_T\delta < \varepsilon$ and so

$$
v(\hat t, X_{\hat t}) \leq 2\varepsilon + J(\hat t, X_{\hat t}, \alpha^{\varepsilon,i})\,.
$$

Therefore RHS of (3.1)

$$
\begin{aligned}
\leq 3\varepsilon + \mathbb{E}\Bigg[&\int_t^{\hat t} f\big(s, X^{\alpha^\varepsilon,t,x}_s, \alpha^\varepsilon_s\big)\, ds\\
&+ \mathbb{E}\left[\int_{\hat t}^T f(s, Y^{\alpha^{\varepsilon,i}}_s, \alpha^{\varepsilon,i}_s)\, ds + g\left(Y^{\alpha^{\varepsilon,i}}_T\right)\Big|Y^{\alpha^{\varepsilon,i}}_{\hat t} = X^{\alpha^\varepsilon,t,x}_{\hat t}\right]\Big|X^{\alpha^\varepsilon,t,x}_s = x\Bigg].
\end{aligned}
$$

Regarding controls we now have the following: $\alpha^\varepsilon \in \mathcal{U}[t,\hat{t}]$ and for each i we have $\alpha^{\varepsilon,i} \in \mathcal{U}(\hat{t},T]$. From these we build one control process β^ε as follows:

$$\beta^\varepsilon_s := \begin{cases} \alpha^\varepsilon_s & s \in [t,\hat{t}] \\ \alpha^{\varepsilon,i}_s & s \in (\hat{t},T] \text{ and } X^{\alpha^\varepsilon,t,x}_{\hat{t}} \in Q_i. \end{cases}$$

This process is progressively measurable with values in A and so $\beta^\varepsilon \in \mathcal{U}[t,T]$. Due to the flow property we may write that RHS of (3.1)

$$\leq 3\varepsilon + \mathbb{E}\left[\int_t^{\hat{t}} f\big(s, X^{\beta^\varepsilon,t,x}_s, \beta^\varepsilon_s\big)\, ds + \int_{\hat{t}}^T f(s, X^{\beta^\varepsilon}_s, \beta^\varepsilon_s)\, ds + g\left(X^{\beta^\varepsilon}_T\right) \Big| X^{\beta^\varepsilon,t,x}_s = x\right].$$

Finally taking supremum over all possible control strategies we see that RHS of (3.1) $\leq 3\varepsilon + v(t,x)$. Letting $\varepsilon \to 0$ completes the proof. $\square$

Lemma 3.2 ($\frac{1}{2}$-Hölder continuity of value function in time)**.** *Let Assumptions 2.3 and 2.5 hold. Then there is a constant $C_T > 0$ such that for any $x \in \mathbb{R}^d$, $0 \leq t, \hat{t} \leq T$ we have*

$$|v(t,x) - v(\hat{t},x)| \leq C_T |t - \hat{t}|^{1/2}.$$

Proof. We will write this proof in the simple case $f = 0$, the general case is left as an exercise. From the Bellman principle we have that

$$v(t,x) = \sup_{\alpha \in \mathcal{U}} \mathbb{E}\left[v(\hat{t}, \left(X^{\alpha,t,x}_{\hat{t}}\right)\right].$$

Then for any $\varepsilon > 0$ we have α^ε such that

$$\begin{aligned} v(t,x) - \varepsilon &\leq \mathbb{E}\left[v\left(\hat{t}, X^{\alpha^\varepsilon,t,x}_{\hat{t}}\right)\right] = \mathbb{E}\left[v\left(\hat{t}, X^{\alpha^\varepsilon,t,x}_{\hat{t}}\right) - v\left(\hat{t}, X^{\alpha^\varepsilon,t,x}_{t}\right) + v\left(\hat{t}, X^{\alpha^\varepsilon,t,x}_{t}\right)\right] \\ &\leq \mathbb{E}\left[\left|v\left(\hat{t}, X^{\alpha^\varepsilon,t,x}_{\hat{t}}\right) - v\left(\hat{t}, X^{\alpha^\varepsilon,t,x}_{t}\right)\right|\right] + v\left(\hat{t}, x\right), \end{aligned}$$

since $X^{\alpha^\varepsilon,t,x}_t = x$. We can now use Lemma 2.8 giving Lipschitz continuity in x of the value function to see that

$$v(t,x) - v(\hat{t},x) - \varepsilon \leq C_T \mathbb{E}\left[\left|X^{\alpha^\varepsilon,t,x}_{\hat{t}} - X^{\alpha^\varepsilon,t,x}_{t}\right|\right].$$

Hölder's inequality and stochastic continuity of solutions to SDEs leads to

$$v(t,x) - v(\hat{t},x) - \varepsilon \leq C_T \left(\mathbb{E}\left[\left|X^{\alpha^\varepsilon,t,x}_{\hat{t}} - X^{\alpha^\varepsilon,t,x}_{t}\right|^2\right]\right)^{1/2} \leq C_T |t - \hat{t}|^{1/2}.$$

We let $\varepsilon \to 0$ to see that $v(t,x) - v(\hat{t},x) \leq C_T |t-\hat{t}|^{1/2}$. On the other hand we have, due to Bellman principle and the Lipschitz continuity in x of the value function, Hölder's inequality and finally stochastic continuity of solutions to SDEs, that

$$\begin{aligned} v(\hat{t},x) - v(t,x) &\leq v(\hat{t},x) - \mathbb{E}\left[v\left(\hat{t}, X^{\alpha^\varepsilon,t,x}_{\hat{t}}\right)\right] \leq \mathbb{E}\left[\left|v(\hat{t},x) - v\left(\hat{t}, X^{\alpha^\varepsilon,t,x}_{\hat{t}}\right)\right|\right] \\ &\leq C_T \mathbb{E}\left[\left|x - X^{\alpha^\varepsilon,t,x}_{\hat{t}}\right|\right] \leq C_T \left(\mathbb{E}\left[\left|x - X^{\alpha^\varepsilon,t,x}_{\hat{t}}\right|^2\right]\right)^{1/2} \leq C_T |t - \hat{t}|^{1/2}. \end{aligned}$$

$\square$

Corollary 3.3. *Let Assumptions 2.3 and 2.5 hold. Then there is a constant $C_T > 0$ such that for any $x, y \in \mathbb{R}^d$, $0 \le s, t \le T$ we have*

$$|v(s,x) - v(t,y)| \le C_T \left(|t - \hat{t}|^{1/2} + |x - y| \right) .$$

This means that the value function v is jointly measurable in (t, x). With this we get the following.

Theorem 3.4 (Bellman's principle / Dynamic programming with stopping time)**.** *For any stopping times $t, \hat{t}$ such that $0 \le t \le \hat{t} \le T$, for any $x \in \mathbb{R}^m$, we have* (3.1).

The proof uses the same arguments as before except that our cubes Q_i now have to cover the whole $[0, T] \times \mathbb{R}^d$ and we need to use the $\frac{1}{2}$-Hölder continuity in time as well.

Corollary 3.5 (Global optimality implies optimality from any time)**.** *Take $x \in \mathbb{R}$. A control $\beta \in \mathcal{U}[0, T]$ is optimal for (P) with the state process $X_s = X_s^{\beta,0,x}$ for $s \in [0, T]$ if and only if for any $\hat{t} \in [0, T]$ we have*

$$v(\hat{t}, X_{\hat{t}}) = J(\hat{t}, X_{\hat{t}}, \beta) .$$

Proof. To ease the notation we will take $f = 0$. The reader is encouraged to prove the general case.

Due to the Bellman principle, Theorem 3.4, we have

$$v(0,x) = \sup_{\alpha \in \mathcal{U}[0,\hat{t}]} \mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}}^{\alpha,0,x} \right) \right] \ge \mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}}^{\beta,0,x} \right) \right] .$$

If β is an optimal control

$$\mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}} \right) \right] \le v(0,x) = J(0,x,\beta) = \mathbb{E}\left[g\left(X_T \right) \right] .$$

Using the tower property of conditional expectation

$$v(0,x) \le \mathbb{E}\left[\mathbb{E}\left[g\left(X_T \right) \middle| \mathcal{F}_{\hat{t}}^X \right] \right] = \mathbb{E}\left[J\left(\hat{t}, X_{\hat{t}}, \beta \right) \right] \le \mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}} \right) \right] \le v(0,x) .$$

Since the very left and very right of these inequalities are equal we get that

$$\mathbb{E}\left[J\left(\hat{t}, X_{\hat{t}}, \beta \right) \right] = \mathbb{E}\left[v\left(\hat{t}, X_{\hat{t}} \right) \right]$$

Moreover $v \ge J$ and so we can conclude that $v\left(\hat{t}, X_{\hat{t}} \right) = J\left(\hat{t}, X_{\hat{t}}, \beta \right)$ a.s. The completes the first part of the proof. The "only if" part of the proof is clear because we can take $\hat{t} = 0$ and get $v(0,x) = J(0,x,\beta)$ which means that β is an optimal control. $\square$

From this observation we can prove the following description of optimality.

Theorem 3.6 (Martingale optimality)**.** *Let the assumptions required for Bellman's principle hold. Fix any initial state x at time $t = 0$ and let*

$$M_t := \int_0^t f\big(s, X_s^{\alpha,0,x}, \alpha(s)\big)\, ds + v\big(t, X_t^{\alpha,0,x}\big) . \tag{3.2}$$

Then for any control $\alpha \in \mathcal{U}[0, T]$ the process $(M_t)_{t \in [0,T]}$ is an $\mathcal{F}_t^X := \sigma(X_s^{\alpha,0,x}; 0 \le s \le t)$ super-martingale. Moreover α is optimal if and only if it is a martingale.

When comparing the subsequent argument to the deterministic case, note how "super-martingale" and "martingale" arise as the stochastic analogues of "decreasing" and "constant", respectively.

Proof. We have by, Theorem 3.1 (the Bellman principle) that for any $0 \le t \le \hat{t} \le T$

$$v\big(t, X_t^{\alpha,0,x}\big) = \sup_{\alpha\in\mathcal{U}} \mathbb{E}\left[\int_t^{\hat{t}} f\big(s, X_s^{\alpha,0,x}, \alpha_s\big)\, ds + v\big(\hat{t}, X_{\hat{t}}^{\alpha,0,x}\big)\Big|\mathcal{F}_t^X\right].$$

Hence for any particular $\alpha \in \mathcal{U}$ we have

$$v\big(t, X_t^{\alpha,0,x}\big) \ge \mathbb{E}\left[\int_t^{\hat{t}} f\big(s, X_s^{\alpha,0,x}, \alpha_s\big)\, ds + v\big(\hat{t}, X_{\hat{t}}^{\alpha,0,x}\big)\Big|\mathcal{F}_t^X\right]$$

and so

$$\begin{aligned} M_t &\ge \int_0^t f\big(s, X_s^{\alpha,0,x}, \alpha(s)\big)\, ds + \mathbb{E}\left[\int_t^{\hat{t}} f\big(s, X_s^{\alpha,0,x}, \alpha_s\big)\, ds + v\big(\hat{t}, X_{\hat{t}}^{\alpha,0,x}\big)\Big|\mathcal{F}_t^X\right] \\ &= \mathbb{E}\left[M_{\hat{t}}|\mathcal{F}_t^X\right] . \end{aligned}$$

This means that M is a super-martingale. Moreover we see that if α is optimal then the inequalities above are equalities and hence M is a martingale.

Now assume that $M_t = \mathbb{E}[M_{\hat{t}}|\mathcal{F}_t^X]$. We want to ascertain that the control α driving M is an optimal one. But the martingale property implies that $J(0,x,\alpha) = \mathbb{E}[M_T] = \mathbb{E}[M_0] = v(0,x)$ and so α is indeed an optimal control. □

One question you may ask yourself is: How can we use the dynamic programming principle to compute an optimal control? Remember that the idea behind the DPP is that it is not necessary to optimize the control α over the entire time interval $[0,T]$ at once; we can partition the time interval into smaller sub-intervals and optimize over each individually. We will see below that this idea becomes particularly powerful if we let the partition size go to zero: the calculation of the optimal control then becomes a pointwise minimization linked to certain PDEs (see Theorem 1.33). That is, for each fixed state x we compute the optimal value of control, say $a \in A$, to apply whenever $X(t) = x$.

3.2 Hamilton-Jacobi-Bellman (HJB) and verification

If the value function $v = v(t,x)$ is smooth enough, then we can apply Itô's formula to v and X in (3.2). Thus we get the *Hamilton-Jacobi-Bellman* (HJB) equation (also know and the *Dynamic Programming equation* or *Bellman* equation).

For notational convenience we will write $\sigma^a(t,x) := \sigma(t,x,a)$, $b^a(t,x) := b(t,x,a)$ and $f^a(t,x) := f(t,x,a)$. We then define

$$L^a v := \frac{1}{2}\sigma^a(\sigma^a)^*\partial_{xx}v + b^a\partial_x v\,.$$

Theorem 3.7 (Hamilton-Jacobi-Bellman (HJB))**.** *If the value function v for (P) is $C^{1,2}$, then it satisfies*

$$\begin{aligned} \partial_t v + \sup_{a\in A}\Big(L^a v + f^a\Big) &= 0 \quad \text{on } [0,T)\times\mathbb{R}^d \\ v(T,x) &= g(x) \quad \forall x\in\mathbb{R}^d\,. \end{aligned} \tag{3.3}$$

Proof. Let $x \in \mathbb{R}$ and $t \in [0,T]$ and assume that v is a $C^{1,2}$ value function for (P). Then the condition $v(T,x) = g(x)$ follows directly from the definition of v. Fix $\alpha \in \mathcal{U}[t,T]$ and let M be given by (3.2). Then, Itô's formula applied to v and X yields

$$dM_s = \left[\left(\partial_t v + L^{\alpha_s} v + f^{\alpha_s}\right)\left(s, X_s^{\alpha,t,x}\right)\right] ds + \left[(\partial_x v\, \sigma^{\alpha_s})\left(s, X_s^{\alpha,t,x}\right)\right] dW_s.$$

For any $(t,x) \in [0,T] \times \mathbb{R}$ take the stopping time $\tau = \tau^{\alpha,t,x}$ given by

$$\tau := \inf\left\{t' \geq t : \int_t^{t'} \left((\partial_x v\, \sigma^{\alpha_s})\left(s, X_s^{\alpha,t,x}\right)\right)^2 ds \geq 1\right\}.$$

We know from Theorem 3.6 that M must be a supermartingale. On the other hand the term given by the stochastic integral is a martingale (when cosidered stopped at τ). So $(M_{t\wedge\tau})_t$ can only be a supermartingale if

$$f^{\alpha_s}(s, X_s) + (\partial_t v + L^{\alpha_s} v)(s, X_s) \leq 0\,.$$

Since the starting point (t,x) and control α were arbitrary we get that we must have

$$(\partial_t v + L^a v + f^a)(t,x) \leq 0 \quad \forall t, x, a\,.$$

Taking the supremum over $a \in A$ we get

$$\partial_t v(t,x) + \sup_{a\in A}[(L^a v + f^a)(t,x)] \leq 0 \quad \forall t, x\,.$$

We now need to show that in fact the inequality cannot be strict. We proceed by setting up a contradiction. Assume that there is (t_0, x_0) such that

$$\partial_t v(t_0, x_0) + \sup_{a\in A}[(L^a v + f^a)(t_0, x_0)] < 0\,.$$

We will show that this contradicts the Bellman principle and hence we must have equality, thus completing the proof.

We must further assume that b and σ are right-continuous in t uniformly in the x variable[4]. Now by continuity (recall that $v \in C^{1,2}$) we get that there must be $\varepsilon > 0$ and an associated $\delta > 0$ such that

$$\partial_t v + \sup_{a\in A}[(L^a v + f^a)] \leq -\varepsilon < 0 \quad \text{on } [t_0 + \delta) \times B_\delta(x_0).$$

Let us fix $\alpha \in \mathcal{U}[t_0, T]$ and let $X_s := X_s^{\alpha,t_0,x_0}$. We define the stopping time

$$\tau := \{s > t_0 : |X_s - x_0| > \delta\} \wedge (t_0 + \delta)\,.$$

Since the process X_s has a.s. continuous sample paths we get $\mathbb{E}[\tau - t_0] > 0$. Let

$$Y_t := \int_{t_0}^t f^{\alpha_s}(s, X_s) ds + v(t, X_t)\,.$$

Then

$$\begin{aligned}
Y_\tau &= v(t_0, x_0) + \int_{t_0}^\tau f^{\alpha_s}(s, X_s) ds + v(\tau, X_\tau) - v(t_0, X_{t_0}) \\
&= v(t_0, x_0) + \int_{t_0}^\tau \left[\left(\partial_t v + L^{\alpha_s} v + f^{\alpha_s}\right)(s, X_s)\right] ds + \int_{t_0}^\tau \left[\partial_x v \sigma^{\alpha_s}(s, X_s)\right] dW_s \\
&\leq v(t_0, x_0) - \varepsilon(\tau - t_0) + \int_{t_0}^\tau \left[\partial_x v \sigma^{\alpha_s}(s, X_s)\right] dW_s\,.
\end{aligned}$$

[4]Does this lead to loss of generality?

Now we take conditional expectation $\mathbb{E}_{t_0,x_0} := \mathbb{E}[\cdot|\mathcal{F}^X_{t_0}]$ on both sides of the last inequality, to get

$$\mathbb{E}_{t_0,x_0}\left[\int_{t_0}^{\tau} f^{\alpha_s}(s,X_s)ds + v(\tau,X_\tau)\right] \leq v(t_0,x_0) - \varepsilon\mathbb{E}_{t_0,x_0}\left[\tau - t_0\right]$$

We now can take the supremum over all controls $\nu \in \mathcal{U}_{ad}[t_0,\tau]$ to get

$$\sup_{\alpha\in\mathcal{U}}\mathbb{E}_{t_0,x_0}\left[\int_{t_0}^{\tau} f^{\alpha_s}(s,X_s)ds + v(\tau,X_\tau)\right] \leq v(t_0,x_0) - \varepsilon\mathbb{E}_{t_0,x_0}\left[\tau - t_0\right]$$

which contradicts the Bellman principle:

$$v(t_0,x_0) = \sup_{\alpha\in\mathcal{U}}\mathbb{E}_{t_0,x_0}\left[\int_{t_0}^{\tau} f^{\alpha_s}(s,X_s)ds + v(\tau,X_\tau)\right].$$

□

Theorem 3.8 (HJB verification). *If, on the other hand, some u in $C^{1,2}$ satisfies* (3.3) *and we have that for all $(t,x)\in[0,T]\times\mathbb{R}$ there is some measurable*

$$a(t,x) \in \arg\max_{a\in A}\Big((L^a u)(t,x) + f^a(t,x)\Big), \tag{3.4}$$

and if

$$dX^*_s = b\big(s,X^*_s,a(s,X^*_s)\big)\,ds + \sigma\big(s,X^*_s,a(s,X^*_s)\big)\,dW_s, \quad X^*_t = x$$

admits a unique solution, and if the process

$$t' \mapsto \int_t^{t'} \partial_x u\big(s,X^*_s\big)\sigma\big(s,X^*_s,a(s,X^*_s)\big)\,dW_s \tag{3.5}$$

is a martingale in $t'\in[t,T]$, then

$$\alpha^*_s := a\big(s,X^*_s\big) \qquad s\in[t,T]$$

is optimal for problem (P) and $v(t,x) = u(t,x)$.

Proof. From Itô's formula applied to v and X^* we can check that for M given by (3.2) (with the control process α^*) we get

$$dM_t = \left[\big(\partial_x v\,\sigma^{\alpha^*_t}\big)\big(t,X^*_t\big)\right]dW_t$$

by using (3.3). This means that M^*_t is a martingale in $t\in[0,T]$. Theorem 3.6 then implies that α^* is optimal. □

Theorem 3.8 is referred as the *verification theorem*. This is key for solving the control problem: if we know the value function v, then the dynamic optimization problem turns into a of static optimization problems at each point (t,x). Recall that (3.4) is calculated pointwise over (t,x).

Exercise 3.9. Find the HJB equation for the following problem. Let $d = 1$, $U = [\sigma_0, \sigma^1] \subset (0,\infty)$, and $k \in \mathbb{R}$. The dynamics of X are given by

$$\frac{dX_s^\nu}{X_s^\nu} = k\,ds + \nu_s\,dW_s,$$

and the value function is

$$v(t,x) = \sup_{\nu\in\mathcal{U}[t,T]} \mathbb{E}[e^{k(t-T)}g\big(X_T^{\nu,t,x}\big)] = -\inf_{\nu\in\mathcal{U}[t,T]} \mathbb{E}[-e^{k(t-T)}g\big(X_T^{\nu,t,x}\big)].$$

This can be interpreted as the pricing equation for an uncertain volatility model with constant interest rate k. The equation is called Black–Scholes–Barenblatt equation and the usual way to present this problem is through a maximization problem.

3.3 Verification and the solving scheme by HJB equation

Theorem 3.7 provides an approach to find optimal solutions:

1. Solve the HJB equation (3.3) (this is typically done by taking a lucky guess and in fact is rarely possible with pen and paper).
2. Find the optimal Markovian control rule $a(t,x)$ calculating (3.4).
3. Solve the optimal control and its state process (u^*, X^*).
4. Verify the martingale condition.

This approach may end up with *failures*. Step one is to solve a fully non-linear second order PDE, that may not have a solution, may have a unique solution or many solutions. If we can prove before hand that the value function for (P) is v is $C^{1,2}$, then the HJB equation admits at least one solution according to Theorem 3.7. The question of uniqueness remains.

In step two, given u that solves (3.3), the problem is a static optimization problem. This is generally much easier to solve.

If we can reach step three, then this step heavily depends on functions b and σ, for which we usually check case by case.

Example 3.10 (Merton problem with power utility and no consumption)**.** This is the classic finance application. The problem can be considered with multiple risky assets but we focus on the situation from Section 2.1.

Recall that we have risk-free asset B_t, risky asset S_t and that our portfolio has wealth given by

$$dX_s = X_s(\nu_s(\mu - r) + r)\,ds + X_s\nu_s\sigma\,dW_s\,,\ \ s\in[t,T]\,,\ \ X_t = x > 0\,.$$

Here ν_s is the control and it describes the fraction of our wealth invested in the risky asset. This can be negative (we short the stock) and it can be more than one (we borrow money from the bank and invest more than we have in the stock).

We take $g(x) := x^\gamma$ with $\gamma \in (0,1)$ a constant. Our aim is to maximize $J^\nu(t,x) := \mathbb{E}^{t,x}\left[g(X_T^\nu)\right]$. Thus our value function is

$$v(t,x) = \sup_{\nu\in\mathcal{U}} J^\nu(t,x) = \sup_{\nu\in\mathcal{U}} \mathbb{E}^{t,x}\left[g(X_T^\nu)\right]\,.$$

This should satisfy the HJB equation (Bellman PDE)

$$\partial_t v + \sup_a \left[\frac{1}{2}\sigma^2 a^2 x^2 \partial_{xx} v + x[(\mu - r)a + r]\partial_x v\right] = 0 \quad \text{on } [0,T) \times (0,\infty)$$
$$v(T,x) = g(x) = x^\gamma \;\; \forall x > 0\,.$$

At this point our best chance is to guess what form the solution may have. We try $v(t,x) = \lambda(t)x^\gamma$ with $\lambda = \lambda(t) > 0$ differentiable and $\lambda(T) = 1$. This way at least the terminal condition holds. If this is indeed a solution then (using it in HJB) we have

$$\lambda'(t) + \sup_a \left[\frac{1}{2}\sigma^2 a^2 \gamma(\gamma - 1) + (\mu - r)\gamma a + r\gamma\right]\lambda(t) = 0 \;\; \forall t \in [0,T)\,, \;\; \lambda(T) = 1\,.$$

since $x^\gamma > 0$ for $x > 0$ and thus we were allowed to divide by this. Moreover we can calculate the supremum by observing that it is quadratic in a with negative leading term $(\gamma - 1)\gamma < 0$. Thus it is maximized when $a^* = \frac{\mu - r}{\sigma^2(1-\gamma)\gamma}$. The maximum itself is

$$\beta(t) := \frac{1}{2}\sigma^2 (a^*)^2 \gamma(\gamma - 1) + (\mu - r)\gamma a^* + r\gamma\,.$$

Thus

$$\lambda'(t) = -\beta(t)\lambda(t)\,, \;\; \lambda(T) = 1 \quad \Longrightarrow \lambda(t) = \exp\left(\int_t^T \beta(s)\,ds\right)\,.$$

Thus we think that the value function and the optimal control are

$$v(t,x) = \exp\left(\int_t^T \beta(s)\,ds\right) x^\gamma \;\text{ and }\; a^* = \frac{\mu - r}{\sigma^2(1-\gamma)\gamma}\,.$$

This now needs to be verified using Theorem 3.8. First we note that the SDE for X^* always has a solution if a^* is a constant.

Next we note that $\partial_x v(s, X_s^*) = \gamma\lambda(s)(X_s^*)^{\gamma-1}$. From Itô's formula

$$\begin{aligned} dX_s^{\gamma-1} &= (\gamma - 1)X_s^{\gamma-2} dX_s + \frac{1}{2}(\gamma - 1)(\gamma - 2)X_s^{\gamma-3} dX_s dX_s \\ &= X_s^{\gamma-1}\left[(\gamma - 1)[a^*(\mu - r) + r]\,ds + \frac{1}{2}(\gamma - 1)(\gamma - 2)a^*\sigma\,dW_s\right]\,. \end{aligned}$$

We can either solve this (like for geometric brownian motion) or appeal to Proposition 1.28 to see that a solution will have all moments uniformly bounded in time on $[0,T]$. Moreover $\lambda = \lambda(t)$ is continuous on $[0,T]$ and thus bounded and so

$$\int_0^T \mathbb{E}\left[\lambda^2(t)|(X_s^*)^{\gamma-1} a^* \sigma X_s^*|^2\right]\,ds < \infty$$

which means that the required expression is a true martingale. This completes verification and Theorem 3.8 gives the conclusion that v is indeed the value function and a^* is indeed the optimal control.

Example 3.11 (Linear-quadratic control problem)**.** This example is a classic engineering application. Note that it can be considered in multiple spatial dimensions but here we focus on the one-dimensional case for simplicity. The multi-dimensional version is e.g. in [Øks00, Ch. 11].

We consider

$$dX_s = [H(s)X_s + M(s)\alpha_s]\, ds + \sigma(s)dW_s\,, s \in [t,T]\,, X_t = x\,.$$

Our aim is to maximize

$$J^\alpha(t,x) := \mathbb{E}^{t,x}\left[\int_t^T (C(s)X_s^2 + D(s)\alpha_s^2)\, ds + RX_T^2\right]\,,$$

where $C = C(t) \leq 0$, $R \leq 0$ and $D = D(t) < -\delta < 0$ are given and deterministic and bounded in t and $\delta > 0$ a constant. The interpretation is the following: since we are losing money at rate C proportionally to X^2, our aim is to make X^2 as small as possible as fast as we can. However controlling X costs us at a rate D proportionally to the strength of control we apply.

The value function is $v(t,x) := \sup_\alpha J^\alpha(t,x)$.

Let us write down the Bellman PDE (HJB equation) we would expect the value function to satisfy:

$$\partial_t v + \sup_a \left[\frac{1}{2}\sigma^2 \partial_x^2 v + [H\, x + M\, a]\partial_x v + C\, x^2 + D\, a^2\right] = 0 \;\text{ on } [0,T)\times\mathbb{R}\,,$$
$$v(T,x) = Rx^2 \;\; \forall x \in \mathbb{R}\,.$$

Since the terminal condition is $g(x) = Rx^2$ let us try $v(t,x) = S(t)x^2 + b(t)$ for some differentiable S and b. We re-write the HJB equation in terms of S and b: (omitting time dependence in H, M, σ, C and D), for $(t,x) \in [0,T)\times\mathbb{R}$,

$$S'(t)x^2 + b'(t) + \sigma^2 S(t) + 2H\, S(t)\, x^2 + C\, x^2 + \sup_a \left[2M(t)\, a\, S(t)\, x + D\, a^2\right] = 0\,,$$
$$S(T) = R \;\text{ and }\; b(T) = 0\,.$$

For fixed t and x we can calculate $\sup_a[2M(t)aS(t)x + D(t)a^2]$ and hence write down the optimal control function $a^* = a^*(t,x)$. Indeed since $D < 0$ and since the expression is quadratic in a we know that the maximum is reached with

$$a^*(t,x) = -(D^{-1}M\, S)(t)\, x\,.$$

We substitute a^* back in to obtain ODEs for $S = S(t)$ and $b = b(t)$ from the HJB equation.

$$\left[S'(t) + 2H\, S(t) + C - D^{-1}M^2S^2(t)\right] x^2 + b'(t) + \sigma^2 S(t) = 0\,,$$
$$S(T) = R \;\text{ and }\; b(T) = 0\,.$$

We collect terms in x^2 and terms independent of x and conclude that this can hold only if

$$S'(t) = D^{-1}M^2S^2(t) - 2H\, S(t) - C\,,\;\; S(T) = R$$

and

$$b'(t) = -\sigma^2 S(t)\,,\;\; b(T) = 0\,.$$

The ODE for S is the *Riccati equation* which has unique solution for $S(T) = R$. We can obtain the expression for $b = b(t)$ by simply integrating:

$$b(T) - b(t) = -\int_t^T \sigma^2(r)S(r)\, dr\,.$$

Then

$$\alpha^*(t,x) = -(D^{-1}MS)(t)x \;\text{ and }\; v(t,x) = S(t)x^2 + \int_t^T \sigma^2(r)S(r)\,dr \tag{3.6}$$

and we see that the control function is measurable. We will now check conditions of Theorem 3.8. The SDE with the optimal control is

$$dX_s^* = \rho(s)X_s^*\,ds + \sigma(s)dW_s\,,\; s \in [t,T]\,,\;\; X_t^* = x\,,$$

where $\rho := H - D^{-1}\,M^2\,S$. This is deterministic and bounded in time. The SDE thus satisfies the Lipschitz conditions and it has a unique strong solution for any t, x.

Since $\partial_x v(r, X_r^*) = 2S(r)X_r^*$, since $\sup_{r\in[t,T]} S^2(r)$ is bounded (continuous function on a closed interval) and since $\sup_{r\in[t,T]} \mathbb{E}[|X_r^*|^2] < \infty$ (moment estimate for SDEs with Lipschitz coefficients) we get

$$\mathbb{E}\int_t^T |S(r)|^2|X_r^*|^2\,dr < \infty$$

and thus conclude that $s \mapsto \int_t^s S(r)X_r^*\sigma(r)\,dW_r$ is a martingale. Thus Theorem 3.8 tells us that the value function and control given by (3.6) are indeed optimal.

3.4 Exercises

Exercise 3.12 (Unattainable optimizer)**.** Here is a simple example in which no optimal control exists, in a finite horizon setting, $T \in (0,\infty)$. Suppose that the state equation is

$$dX_t = \alpha_t\,dt + \,dW_t, \qquad X_0 = x \in \mathbb{R}.$$

A control α is admissible ($\alpha \in \mathcal{U}$) if: α takes values in $\mathbb{R}$, is $\mathcal{F}_t$-prog.-meas., there exists a unique solution to the state equation and $\mathbb{E}\int_0^T \alpha_s^2\,ds < \infty$.

The value function is $v(t,x) := \inf_{\nu\in\mathcal{U}_{ad}[t,T]} J(t,x,\nu)$. Clearly $v(t,x) \geq 0$.

i) Show that $J^\alpha(t,x) = \mathbb{E}[X_T^2] < \infty$.

ii) Show that if $\alpha_t := -cX_t$ for some constant $c \in (0,\infty)$ then $\alpha \in \mathcal{U}$ and

$$J^\alpha(t,x) = J^{cX}(t,x) = \frac{1}{2c} - \frac{1-2cx^2}{2c}e^{-2c(T-t)}.$$

Hint: with such an α, the process X is an Ornstein-Uhlenbeck process, see Exercise 1.41.

iii) Conclude that $v(t,x) = 0$ for all $t \in [0,T)$, $x \in \mathbb{R}$.

iv) Show that there is no $\alpha \in \mathcal{U}[t,T]$ such that $J(t,x,\alpha) = 0$. *Hint:* Suppose that there is such a α and show that this leads to a contradiction.

v) The associated HJB equation is

$$\partial_t v + \inf_{a\in\mathbb{R}}\left\{\frac{1}{2}\partial_{xx}v + a\partial_x v\right\} = 0, \quad \text{on } [0,T)\times\mathbb{R}.$$
$$v(T,x) = x^2\,.$$

Show that there is no value $\alpha \in \mathbb{R}$ for which the infimum is attained.

Conclusions from Exercise 3.12: The value function $v(t,x) = \inf_{\alpha\in\mathcal{U}} J(t,x,\alpha)$ satisfies $v(t,x) = 0$ for all $(t,x) \in [0,T] \times \mathbb{R}$ but there is no admissible control α which attains the v (i.e. there is no $\alpha^* \in \mathcal{U}$ such that $v(t,x) = J(t,x,\alpha^*)$).

The goal in this problem is to bring the state process as close as possible to zero at the terminal time T. However, as defined above, there is no cost of actually controlling the system. We can set α arbitrarily large without any negative consequences. From a modelling standpoint, there is often a trade-off between costs incurred in applying control and our overall objective. Compare this with Example 3.11.

Exercise 3.13 (Merton problem with exponential utility)**.** We return to the portfolio optimization problem, see Section 2.1. Unlike in Example 3.10 we consider the utility function $g(x) := -e^{-\gamma x}$, $\gamma > 0$ a constant. We will also take $r = 0$ for simplicity and assume there is no consumption ($C = 0$). With X_t denoting the wealth at time time t we have the value function given by

$$v(t,x) = \sup_{\pi\in\mathcal{U}} \mathbb{E}\left[g\left(X_T^{\pi,t,x,}\right)\right].$$

i) Write down the expression for the wealth process in terms of π, the amount of wealth invested in the risky asset and with $r = 0$, $C = 0$.

ii) Write down the HJB equation associated to the optimal control problem. Solve the HJB equation by inspecting the terminal condition and thus suggesting a possible form for the solution. Write down the optimal control explicitly.

iii) Use verification theorem to show that the solution and control obtained in previous step are indeed the value function and optimal control.

Exercise 3.14 ([Sei09, p252, Prob. 4.8])**.** Solve the problem

$$\max_{\nu} \mathbb{E}\big[-\int_0^T \nu^2(t)\frac{e^{-X(t)}}{2}\,dt + e^{X(T)}\big],$$

where ν takes values in $\mathbb{R}$, subject to $dX(t) = \nu(t)e^{-X(t)}\,dt + \sigma\,dW(t)$, $X(0) = x_0 \in \mathbb{R}$, $\sigma \in (0,\infty)$, σ, x_0 are fixed numbers.

Hint: Try a solution of the HJB equation of the form $v(t,x) = \phi(t)e^x + \psi(t)$.

For more exercises, see [Sei09, Exercise 4.13, 4.14, 4.15].

3.5 Solutions to Exercises

Solution (to Exercise 3.12).

i) We use the fact that $\mathbb{E}\int_0^T \alpha_r^2\,dr < \infty$ for admissible control. We also use that $(a+b)^2 \le 2a^2 + 2b^2$. Then for, any $s \in [t,T]$,

$$\mathbb{E}[X_s^2] \le 4x^2 + 4\mathbb{E}\left(\int_t^s \alpha_r\,dr\right)^2 + 2\mathbb{E}(W_s - W_t)^2\,.$$

With Hölder's inequality we get

$$\mathbb{E}[X_s^2] \le 4x^2 + 4(s-t)^{1/2}\mathbb{E}\int_t^s \alpha_r^2\,dr + 2(s-t) \le c_T\left(1 + x^2 + \mathbb{E}\int_0^T \alpha_r^2\,dr\right) < \infty\,. \tag{3.7}$$

ii) Substitute $\alpha_s = -cX_s$. The Ornstein-Uhlenbeck SDE, see Exercise 1.41, has solution

$$X_T = e^{-c(T-t)}x + \int_t^T e^{-c(T-t)}\,dW_r\,.$$

We square this, take expectation (noting that the integrand in the stochastic integral is deterministic and square integrable):

$$\mathbb{E}X_T^2 = e^{-2c(T-t)}x^2 + \mathbb{E}\left(\int_t^T e^{-c(T-t)}\,dW_r\right)^2\,.$$

With Itô's isometry we get

$$\mathbb{E}X_T^2 = e^{-2c(T-t)}x^2 + \int_t^T e^{-2c(T-t)}\,dr\,.$$

Now we just need to integrate to obtain $J^\alpha(t,x) = J^{cX}(t,x) = \mathbb{E}X_T^2$.

iii) We know that $v(t,x) \ge 0$ already. Moreover

$$v(t,x) = \inf_{\alpha\in\mathcal{U}} J^\alpha(t,x) \le \lim_{c\nearrow\infty} J^{cX}(t,x) = \lim_{c\nearrow\infty}\left[\frac{1}{2c} - \frac{1-2cx^2}{2c}e^{-2c(T-t)}\right] = 0\,.$$

iv) Assume that an optimal $\alpha^* \in \mathcal{U}$ exists so that $\mathbb{E}[X_T^{\alpha^*,t,x}] = J^{\alpha^*}(t,x) = 0$ for any $t < T$ and any x. We will show this leads to contradiction.
First of all, we can calculate using Itô formula that

$$dX_s^* = 2X_s^*\alpha_s^*\,ds + 2X_s^*\,dW_s + ds\,.$$

Hence

$$0 = \mathbb{E}[(X_T^*)^2] = x^2 + 2\mathbb{E}\int_t^T (X_s^*\alpha_s^* + 1)\,ds + \mathbb{E}\int_t^T X_s^*\,dW_s\,.$$

But since α^* is admissible we have $\int_t^T \mathbb{E}(X_s^*)^2\,ds < \infty$ due to (3.7). This means that the stochastic integral is a martingale and hence its expectation is zero. We now use Fatou's lemma and take the limit as $t \nearrow T$. Then

$$-x^2 = 2\liminf_{t\nearrow T}\mathbb{E}\int_t^T (X_s^*\alpha_s^* + 1)\,ds \ge 2\mathbb{E}\left[\liminf_{t\nearrow T}\int_t^T (X_s^*\alpha_s^* + 1)\,ds\right] = 0\,.$$

So $-x^2 \ge 0$. This cannot hold for all $x \in \mathbb{R}$ and so we have contradiction.

v) If $\partial_x v(t,x) \ne 0$, then $a = \pm\infty$. If $\partial_x V(t,x) = 0$, then a is undefined. One way or another there is no real number attaining the infimum.

Solution (to Exercise 3.13). The wealth process (with the control expressed as π, the amount of wealth invested in the risky asset and with $r = 0$, $C = 0$), is given by

$$dX_s = \pi_s\mu\,ds + \pi_s\sigma\,dW_s\,,\quad s \in [t,T]\,,\quad X_t = x > 0\,. \tag{3.8}$$

The associated HJB equation is

$$\partial_t v + \sup_{p\in\mathbb{R}}\left[\frac{1}{2}p^2\sigma^2\partial_{xx}v + p\,\mu\,\partial_x v\right] = 0 \text{ on } [0,T)\times\mathbb{R},$$
$$v(T,x) = g(x)\ \ \forall x \in \mathbb{R}\,.$$

We make a guess that $v(t,x) = \lambda(t)g(x) = -\lambda(t)e^{-\gamma x}$ for some differentiable function $\lambda = \lambda(t) \geq 0$. Then, since we can divide by $-e^{-\gamma x} \neq 0$ and since we can factor out the non-negative $\lambda(t)$, the HJB equation will hold provided that

$$\lambda'(t) + \sup_{p\in\mathbb{R}} \left[-\frac{1}{2} p^2 \sigma^2 \gamma^2 + p\,\mu\,\gamma \right] \lambda(t) = 0 \text{ on } [0,T),\, \lambda(T) = 1.$$

The supremum is attained for $p^* = \frac{\mu}{\sigma^2\gamma}$ since the expression we are maximizing is quadratic in p with negative leading order term. Thus $\lambda'(t) + \beta(t)\lambda(t) = 0$ and $\lambda(T) = 1$ with

$$\beta(t) := -\frac{1}{2}(p^*)^2\sigma^2\gamma^2 + p^*\,\mu\,\gamma = -\frac{1}{2}\mu\gamma + \frac{\mu^2}{\sigma^2}\,.$$

We can solve the ODE for λ to obtain

$$\lambda(t) = e^{\int_t^T \beta(r)\,dr}$$

and hence our candidate value function and control are

$$v(t,x) = e^{\int_t^T \beta(r)\,dr} g(x) \text{ and } p^* = \frac{\mu}{\sigma^2\gamma}\,.$$

We now need to use Theorem 3.8 to be able to confirm that these are indeed the value function and optimal control.

First of all the solution for optimal X^* always exists since we just need to integrate in the expression (3.8) taking $\pi_t := p^*$. We note that the resulting process is Gaussian.

Now $\partial_x v(s, X_s^*) = \lambda(t)\,\gamma e^{-\gamma X_s^*}$. We can now use what we know about moment generating functions of normal random variables to conclude that

$$\int_t^T \lambda(s)^2\, e^{-2\gamma X_s^*}\, ds < \infty.$$

The process

$$\bar{t} \mapsto \int_t^{\bar{t}} \lambda(s)\, e^{-\gamma X_s^*}\, dW_s$$

is thus a true martingale and the verification is complete.

Solution (to Exercise 3.14).

$$\psi(t) = 0, \qquad \phi(t) = \frac{\sigma^2}{Ce^{\sigma^2 t/2} - 1}, \qquad C = (1+\sigma^2)e^{-\sigma^2 T/2}.$$

4 Maximum Principle and Backward Stochastic Differential Equations

17h14, 29/04/2018

In the previous part, we developed the dynamic programming theory for the stochastic control problem with Markovian system.

We introduce another approach called maximum principle, originally due to Pontryagin in the deterministic case. We will also study this approach to study the control problem (P).

4.1 Backward Stochastic Differential Equations (BSDEs)

For a deterministic differential equation

$$\frac{dx(t)}{dt} = b(t, x(t)) \ \ t \in [0, T] \, , \ \ x(T) = a$$

we can reverse the time by changing variables. Let $\tau := T - t$ and $y(\tau) = x(t)$. Then we have

$$\frac{dy(\tau)}{d\tau} = -b(T - \tau, y(\tau)) \ \ \tau \in [0, T] \, , \ \ y(0) = a \, .$$

So the *backward* ODE is equivalent to a *forward* ODE.

The same argument would fail for SDEs since the time-reversed SDE would not be adapted to the appropriate filtration and the stochastic integrals will not be well defined.

Recall the martingale representation theorem (see Theorem 1.25), which says any $\xi \in L^2_{\mathcal{F}_T}$ can be uniquely represented by

$$\xi = \mathbb{E}[\xi] + \int_0^T \phi_t \, dW_t \, .$$

If we define $M_t = \mathbb{E}[\xi] + \int_0^t \phi_s dW_s$, then M_t satisfies

$$dM_t = \phi_t \, dW_t \, , \ \ M_T = \xi \, .$$

This leads to the idea that a solution to a *backward* SDE must consist of two processes (in the case above M and ϕ).

Consider the backward SDE (BSDE)

$$dY_t = g_t\big(Y_t, Z_t\big) \, dt + Z_t \, dW_t, \qquad Y(T) = \xi \, .$$

We shall give a few examples when this has explicit solution.

Example 4.1. Assume that $g = 0$. In this case, $Y_t = \mathbb{E}[\xi|\mathcal{F}_t]$ and Z is the process given by the martingale representation theorem.

Example 4.2. Assume that $g_t(y, z) = \gamma_t$. In this case, take $\hat{\xi} := \xi - \int_0^T \gamma_t \, dt$. We get the solution $(\hat{Y}, \hat{Z})$ to

$$d\hat{Y}_t = \hat{Z}_t \, dW_t \, , \ \ \hat{Y}_T = \hat{\xi}$$

as

$$\hat{Y}_t = \mathbb{E}\left[\hat{\xi}|\mathcal{F}_t\right] = \mathbb{E}\left[\xi - \int_0^T \gamma_t \, dt \middle| \mathcal{F}_t\right]$$

and we get Z from the martingale representation theorem. Then with $Y_t := \hat{Y}_t + \int_0^t \gamma_s\,ds$, $Z_t := \hat{Z}_t$ we have a solution (Y, Z) so in particular

$$Y_t = \mathbb{E}\left[\xi - \int_0^T \gamma_t\,dt \middle| \mathcal{F}_t\right] + \int_0^t \gamma_s\,ds = \mathbb{E}\left[\xi - \int_t^T \gamma_s\,ds \middle| \mathcal{F}_t\right].$$

Example 4.3. Assume that $g_t(y,z) = \alpha_t y + \beta_t z + \gamma_t$ and $\alpha = \alpha_t$, $\beta = \beta_t$, $\gamma = \gamma_t$ are adapted processes that satisfy certain integrability conditions (those will become clear). We will construct a solution using an exponential transform and a change of measure.

Consider a new measure $\mathbb{Q}$ given by the Radon–Nikodym derivative

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\int_0^T \beta_s^2\,ds - \int_0^T \beta_s\,dW_s\right)$$

and assume that $\mathbb{E}\left[\frac{d\mathbb{Q}}{d\mathbb{P}}\right] = 1$. Then, due to Girsanov's Theorem 1.24, the process given by $W_t^{\mathbb{Q}} = W_t + \int_0^t \beta_s\,ds$ is a $\mathbb{Q}$-Wiener process. Consider the BSDE

$$d\bar{Y}_t = \bar{\gamma}_t\,dt + \bar{Z}_t\,dW_t^{\mathbb{Q}}\,,\ \ \bar{Y}_T = \bar{\xi}\,, \tag{4.1}$$

where $\bar{\gamma}_t := \gamma_t \exp\left(-\int_0^t \alpha_s\,ds\right)$ and $\bar{\xi} := \xi \exp\left(-\int_0^T \alpha_s\,ds\right)$. We know from Example 4.2 that this BSDE has a solution $(\bar{Y}, \bar{Z})$ and in fact we know that

$$\bar{Y}_t = \mathbb{E}^{\mathbb{Q}}\left[\xi e^{-\int_0^T \alpha_s\,ds} - \int_t^T \gamma_s e^{-\int_0^s \alpha_r\,dr}\,ds \middle| \mathcal{F}_t\right].$$

We let $Y_t := \bar{Y}_t \exp\left(\int_0^t \alpha_s\,ds\right)$ and $Z_t := \bar{Z}_t \exp\left(\int_0^t \alpha_s\,ds\right)$. Now using the Itô product rule with (4.1) and the equation for $W^{\mathbb{Q}}$ we can check that

$$\begin{aligned} dY_t &= d\left(\bar{Y}_t e^{\int_0^t \alpha_s\,ds}\right) = \alpha_t Y_t\,dt + e^{\int_0^t \alpha_s\,ds}\,d\bar{Y}_t = \alpha_t Y_t\,dt + \gamma_t\,dt + Z_t\,dW_t^{\mathbb{Q}} \\ &= (\alpha_t Y_t + \beta_t Z_t + \gamma_t)\,dt + Z_t\,dW_t \end{aligned}$$

and moreover $Y_T = \xi$. In particular we get

$$Y_t = \mathbb{E}^{\mathbb{Q}}\left[\xi e^{-\int_t^T \alpha_s\,ds} - \int_t^T \gamma_s e^{-\int_t^s \alpha_r\,dr}\,ds \middle| \mathcal{F}_t\right]. \tag{4.2}$$

To get the solution as an expression in the original measure we need to use the Bayes formula for conditional expectation, see Proposition A.13. We obtain

$$Y_t = \frac{\mathbb{E}\left[\left(\xi e^{-\int_t^T \alpha_s\,ds} - \int_t^T \gamma_s e^{-\int_t^s \alpha_r\,dr}\,ds\right) e^{-\frac{1}{2}\int_0^T \beta_s^2\,ds - \int_0^T \beta_s\,dW_s} \middle| \mathcal{F}_t\right]}{\mathbb{E}\left[e^{-\frac{1}{2}\int_0^T \beta_s^2\,ds - \int_0^T \beta_s\,dW_s} \middle| \mathcal{F}_t\right]}.$$

Proposition 4.4 (Boundedness of solutions to linear BSDEs). *Consider the linear backward SDE with $g_t(y,z) = \alpha_t y + \beta_t z + \gamma_t$. If α, β, γ and ξ are all bounded then the process Y in the solution pair (Y, Z) is bounded.*

Proof. This proof is left as exercise. □

Example 4.5 (BSDE and replication in the Black-Scholes market)**.** In a standard Black-Scholes market model we have a risk-free asset $dB_t = rB_t\,dt$ and risky assets

$$dS_t = \text{diag}(\mu)S_t\,dt + \sigma S_t\,dW_t\,.$$

Here μ is the drift vector of the risky asset rate, σ is the volatility matrix.

Let π denote the cash amount invested in the risky asset and X the replicating portfolio value (so $X - \pi S$ is invested in the risk-free asset). Then the self-financing property says that (interpreting $1/S$ to be $(1/S_1, \dots, 1/S_m)^\top$)

$$dX_t = \pi_t \frac{1}{S_t}\,dS_t + \frac{X_t - \pi_t S_t}{B_t}\,dt$$

i.e.

$$dX_t = \big[rX_t + \pi_t(\mu - r)\big]\,dt + \pi_t^\top \sigma\,dW(t)\,.$$

We can define $Z_t = \sigma_t^\top \pi_t$ and if σ^{-1} exists then $\pi_t = (\sigma^\top)^{-1} Z_t = (\sigma^{-1})^\top Z_t$

$$dX_t = \big[rX_t + (\mu^\top - r)(\sigma^{-1})^\top Z_t\big]\,dt + Z_t\,dW_t.$$

For any payoff ξ at time T, the replication problem is to solve the BSDE given by this differential coupled with $X_T = \xi$. If $\xi \in L^2_{\mathcal{F}_T}$ the equation admits a unique square-integrable solution (X, Z). Hence the cash amount invested in the risky asset, required in the replicating portfolio is $\pi_t = (\sigma^{-1})^\top Z_t$, and the replication cost (contingent claim price) at time t is X_t.

We see that this is a BSDE with linear driver and so from Example 4.3 we have, see (4.2) that

$$X_t = \mathbb{E}^{\mathbb{Q}}\left[\xi e^{-r(T-t)} \middle| \mathcal{F}_t\right]\,,$$

where

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = e^{-\frac{1}{2}|\sigma^{-1}(\mu - r)|^2 T - (\mu^\top - r)(\sigma^{-1})^\top W_T}\,.$$

In other words we see that $\mathbb{Q}$ is the usual risk-neutral measure we get in Black–Scholes pricing.

A standard backward SDE (BSDE) is formulated as

$$dY_t = g_t\big(Y_t, Z_t\big)\,dt + Z_t\,dW_t, \qquad Y(T) = \xi, \tag{4.3}$$

where $g = g_t(\omega, y, z)$ must be such that $g_t(y, z)$ is at least $\mathcal{F}_t$-measurable for any fixed t, y, z. We will refer to g is called as the *generator* or *driver* of the Backward SDE.

Definition 4.6. Given $\xi \in L^2(\mathcal{F}_T)$ and a generator g, a pair of $(\mathcal{F}_t)_{t \in [0,T]}$-adapted processes (Y, Z) is called as a solution for (4.3) if

$$Y_t = \xi - \int_t^T g_s\big(Y_s, Z_s\big)\,ds - \int_t^T Z_s\,dW_s, \quad \forall t \in [0, T].$$

Theorem 4.7 (Existence and uniqueness for BSDEs)**.** *Suppose $g = g_t(y, z)$ satisfies*

(i) We have $g(0, 0) \in \mathcal{H}$.

(ii) There exists a constant $L > 0$ such that

$$|g_t(y,z) - g_t(\overline{y},\overline{z})| \leq L(|y-\overline{y}| + |z-\overline{z}|)\ \text{,a.s.}\ \forall t \in [0,T]\,, \forall y,z,\overline{y},\overline{z}\,.$$

Then for any $\xi \in L^2_{\mathcal{F}_T}$, there exists a unique $(Y,Z) \in \mathcal{H}\times\mathcal{H}$ solving the BSDE (4.3).

Recall that $\mathcal{H}$ is the space introduced in Definition 1.18.

Proof. We consider the map $\Phi = \Phi(U,V)$ for (U,V) in $\mathcal{H}\times\mathcal{H}$. Given (U,V) we define $(Y,Z) = \Phi(U,V)$ as follows. Let $\hat{\xi} := \xi - \int_0^T g_s(U_s,V_s)\,ds$. Then

$$\begin{aligned}\mathbb{E}\int_0^T |g_s(U_s,V_s)|^2\,ds &\leq \mathbb{E}\int_0^T [2|g_s(U_s,V_s) - g_s(0,0)|^2 + 2|g_s(0,0)|^2]\,ds \\ &\leq \mathbb{E}\int_0^T [2L^2(|U_s|^2 + |V_s|^2) + 2|g_s(0,0)|^2]\,ds < \infty,\end{aligned} \tag{4.4}$$

since U and V and $g(0,0)$ are in $\mathcal{H}$. So $\hat{\xi} \in L^2(\mathcal{F}_T)$ and we know that for $\hat{Y}_t := \mathbb{E}[\hat{\xi}|\mathcal{F}_t]$ there is Z such that

$$d\hat{Y}_t = Z_t\,dW_t\,,\ \ \hat{Y}_T = \hat{\xi}\,.$$

Take $Y_t := \hat{Y}_t + \int_0^t g_s(U_s,V_s)\,ds$. Then

$$Y_t = \xi - \int_t^T g_s(U_s,V_s)\,ds - \int_t^T Z_s\,dW_s\,. \tag{4.5}$$

The next step is to show that $(U,V) \mapsto \Phi(U,V) = (Y,Z)$ described above is a contraction on an appropriate Banach space.

We will assume, for now, that $|\xi| \leq N$ and that $|g| \leq N$. We consider (U,V) and (U',V'). From these we obtain $(Y,Z) = \Phi(U,V)$ and $(Y',Z') = \Phi(U',V')$. We will write

$$(\bar{U},\bar{V}) := (U-U', V-V')\,,\ \ (\bar{Y},\bar{Z}) := (Y-Y', Z-Z')\,,\ \ \bar{g} := g(U,V) - g(U',V')\,.$$

Then

$$d\bar{Y}_s = \bar{g}_s\,ds + \bar{Z}_s dW_s$$

and with Itô formula we see that

$$d\bar{Y}_s^2 = 2\bar{Y}_s\bar{g}_s\,ds + 2\bar{Y}_s\bar{Z}_s\,dW_s + \bar{Z}_s^2 ds\,.$$

Hence, for some $\beta > 0$,

$$d(e^{\beta s}\bar{Y}_s^2) = e^{\beta s}\left[2\bar{Y}_s\bar{g}_s\,ds + 2\bar{Y}_s\bar{Z}_s\,dW_s + \bar{Z}_s^2\,ds + \beta\bar{Y}_s^2\,ds\right]\,.$$

Noting that, due to (4.5), we have $\bar{Y}_T = Y_T - Y'_T = 0$, we get

$$0 = \bar{Y}_0^2 + \int_0^T e^{\beta s}\left[2\bar{Y}_s\bar{g}_s + \bar{Z}_s^2 + \beta\bar{Y}_s^2\right]\,ds + \int_0^T 2e^{\beta s}\bar{Y}_s\bar{Z}_s\,dW_s\,.$$

Since $Z \in \mathcal{H}$ we have

$$\mathbb{E}\int_0^T 4e^{2\beta s}|\bar{Y}_s|^2|\bar{Z}_s|^2\,ds \leq e^{2\beta T}4N^2(1+T)^2\mathbb{E}\int_0^T |\bar{Z}_s|^2\,ds < \infty$$

and so, the stochastic integral being a martingale, we get

$$\mathbb{E}\int_0^T e^{\beta s}\left[\bar{Z}_s^2+\beta\bar{Y}_s^2\right]\,ds=-\mathbb{E}\bar{Y}_0^2-\mathbb{E}\int_0^T e^{\beta s}2\bar{Y}_s\bar{g}_s\,ds\le 2\mathbb{E}\int_0^T e^{\beta s}|\bar{Y}_s||\bar{g}_s|\,ds\,.$$

Using the Lipschitz continuity of g and Young's inequality (with $\varepsilon=1/4$) we have

$$\begin{aligned}e^{\beta s}|\bar{Y}_s||\bar{g}_s|&\le e^{\beta s}|\bar{Y}_s|L(|\bar{U}_s|+|\bar{V}_s|)\le 2L^2e^{\beta s}|\bar{Y}_s|^2+\frac{1}{8}e^{\beta s}(|\bar{U}_s|+|\bar{V}_s|)^2\\&\le 2L^2e^{\beta s}|\bar{Y}_s|^2+\frac{1}{4}e^{\beta s}(|\bar{U}_s|^2+|\bar{V}_s|^2)\,.\end{aligned}$$

We can now take $\beta=1+4L^2$ and we obtain

$$\mathbb{E}\int_0^T e^{\beta s}\left[\bar{Z}_s^2+\bar{Y}_s^2\right]\,ds\le\frac{1}{2}\mathbb{E}\int_0^T e^{\beta s}(|\bar{U}_s|^2+|\bar{V}_s|^2)\,ds\,. \tag{4.6}$$

We now need to remove the assumption that $|\xi|\le N$ and $|g|\le N$. To that end consider $\xi^N:=-N\wedge\xi\vee N$ and $g^N:=-N\wedge g\vee N$ (so $|\xi^N|\le N$ and $|g^N|\le N$). We obtain Y^N, Z^N as before. Note that

$$Y_t=\mathbb{E}[\xi|\mathcal{F}_t]=\mathbb{E}\left[\lim_{N\to\infty}\hat{\xi}^N|\mathcal{F}_t\right]=\lim_{N\to\infty}Y_t^N$$

due to Lebesgue's dominated convergence for conditional expectations. Indeed, we have $|\hat{\xi}^N|\le|\xi|+\int_0^T|g_s(U_s,V_s)|\,ds$ and this is in L^2 due to (4.4). Moreover

$$\begin{aligned}\mathbb{E}\int_0^T|Z_t^N-Z_t|^2\,dt&=\mathbb{E}\left(\int_0^T(Z_t^N-Z_t)\,dW_t\right)^2=\mathbb{E}\left(Y_T^N-Y_T+Y_0-Y_0^N\right)^2\\&\le 2\mathbb{E}|Y_T^N-Y_T|^2+2\mathbb{E}|Y_0-Y_0^N|^2\to 0\ \text{ as } N\to\infty\end{aligned}$$

due to Lebesgue's dominated convergence theorem. Then from (4.6) be have, for each N,

$$\mathbb{E}\int_0^T e^{\beta s}\left[|\bar{Z}_s^N|^2+|\bar{Y}_s^N|^2\right]\,ds\le\frac{1}{2}\mathbb{E}\int_0^T e^{\beta s}(|\bar{U}_s|^2+|\bar{V}_s|^2)\,ds\,.$$

But since the RHS is independent of N, we obtain (4.6) but now without the assumption that $|\xi|\le N$ and $|g|\le N$. Consider now the Banach space $(\mathcal{H}\times\mathcal{H},\|\cdot\|)$, with

$$\|(Y,Z)\|:=\mathbb{E}\int_0^T e^{\beta s}\left[Z_s^2+Y_s^2\right]\,ds\,.$$

From (4.6) we have

$$\|\Phi(U,V)-\Phi(U',V')\|\le\frac{1}{2}\|(U,V)-(U',V')\|\,.$$

So the map $\Phi:\mathcal{H}\times\mathcal{H}\to\mathcal{H}\times\mathcal{H}$ is a contraction and due to Banach's Fixed Point Theorem there is a unique (Y^*,Z^*) which solves the equation $\Phi(Y^*,Z^*)=(Y^*,Z^*)$. Hence

$$Y_t^*=\xi-\int_t^T g_s(Y_s^*,Z_s^*)\,ds-\int_t^T Z_s^*\,dW_s$$

due to (4.5). □

Theorem 4.8. *Let (Y^1, Z^1) and (Y^2, Z^2) be solutions to BSDEs with generators and terminal conditions g^1, ξ^1 and g^2, ξ^2 respectively. Assume that $\xi^1 \leq \xi^2$ a.s. and that $g_t^2(Y_t^2, Z_t^2) \leq g_t^1(Y_t^1, Z_t^1)$ a.e. on $\Omega \times (0, T)$. Assume finally that the generators satisfy the assumption of Theorem 4.7 and $\xi^1, \xi^2 \in L^2(\mathcal{F}_T)$. Then $Y^1 \leq Y^2$.*

Proof. We note that the BSDE satisfied by $\bar{Y} := Y^2 - Y^1$, $\bar{Z} := Z^2 - Z^1$ is

$$d\bar{Y}_t = [g_t^2(Y_t^2, Z_t^2) - g_t^1(Y_t^1, Z_t^1)]\, dt + \bar{Z}_t\, dW_t\,, \;\; \bar{Y}_T = \bar{\xi} := \xi^2 - \xi^1\,.$$

This is

$$\begin{aligned} d\bar{Y}_t =& [g_t^2(Y_t^2, Z_t^2) - g_t^2(Y_t^1, Z_t^2) + g_t^2(Y_t^1, Z_t^2) - g_t^2(Y_t^1, Z_t^1) + g_t^2(Y_t^1, Z_t^1) - g_t^1(Y_t^1, Z_t^1)]\, dt \\ & + \bar{Z}_t\, dW_t\,, \;\; \bar{Y}_T = \bar{\xi} \end{aligned}$$

which we can re-write as

$$d\bar{Y}_t = [\alpha_t \bar{Y}_t + \beta_t \bar{Z}_t + \gamma_t]\, dt + \bar{Z}_t\, dW_t\,, \;\; \bar{Y}_T = \bar{\xi}\,,$$

where

$$\alpha_t := \frac{g_t^2(Y_t^2, Z_t^2) - g_t^2(Y_t^1, Z_t^2)}{Y_t^2 - Y_t^1} \mathbb{1}_{Y_t^1 \neq Y_t^2}\,, \;\; \beta_t := \frac{g_t^2(Y_t^1, Z_t^2) - g_t^2(Y_t^1, Z_t^1)}{Z_t^2 - Z_t^1} \mathbb{1}_{Z_t^1 \neq Z_t^2}$$

and where

$$\gamma_t := g_t^2(Y_t^1, Z_t^1) - g_t^1(Y_t^1, Z_t^1)\,.$$

Due to the Lipschitz assumption on g^2 we get that α and β are bounded and since Y^i, Z^i are in $\mathcal{H}$ we get that $\gamma \in \mathcal{H}$. Thus we have an affine BSDE for $(\bar{Y}, \bar{Z})$ and the conclusion follows from (4.2) since we get

$$\bar{Y}_t = \mathbb{E}^{\mathbb{Q}}\left[\underbrace{\bar{\xi} e^{-\int_t^T \alpha_s\, ds}}_{\geq 0} - \underbrace{\int_t^T \gamma_s e^{-\int_t^s \alpha_r\, dr}\, ds}_{\leq 0} \middle| \mathcal{F}_t \right] \geq 0$$

from the assumptions that $\xi^1 \leq \xi^2$ a.s. and that $g_t^2(Y_t^2, Z_t^2) \leq g_t^1(Y_t^1, Z_t^1)$ a.e. □

4.2 Pontryagin's Maximum Principle

We now return to the optimal control problem (P). Recall that given running gain f and terminal gain g our aim is to optimally control

$$dX_t^\alpha = b_t(X_t, \alpha_t)\, dt + \sigma_t(X_t, \alpha_t)\, dW_t, \;\; t \in [0, T]\,, \;\; X_0^\alpha = x,$$

where $\alpha \in \mathcal{U}$ and we assume that Assumption 2.3 holds. Recall that by optimally controlling the process we mean a control which will maximize

$$J(\alpha) := \mathbb{E}\left[\int_0^T f(t, X_t^\alpha, \alpha_t)\, dt + g(X_T^\alpha) \right]$$

over $\alpha \in \mathcal{U}$. Unlike in Chapter 3 we can consider the process starting from time 0 (because we won't be exploiting the Markov property of the SDE) and unlike in Chapter 3 we will assume that A is a subset of $\mathbb{R}^m$.

We define the *Hamiltonian* $H : [0, T] \times \mathbb{R}^d \times A \times \mathbb{R}^d \times \mathbb{R}^{d \times d'} \to \mathbb{R}$ of the system as

$$H_t(x, a, y, z) := b_t(x, a)\, y + \mathrm{tr}[\sigma_t^\top(x, a)\, z] + f_t(x, a)\,.$$

Assumption 4.9. Assume that $x \mapsto H_t(x,a,y,z)$ is differentiable for all a,t,y,z with derivative bounded uniformly in a,t,y,z. Assume that g is differentiable in x with the derivative having at most linear growth (in x).

Consider the *adjoint BSDEs* (one for each $\alpha \in \mathcal{U}$)

$$dY_t^\alpha = -\partial_x H_t(t, X_t, \alpha_t, Y_t^\alpha, Z_t^\alpha)\,dt + Z_t\,dW_t\,, \;\; Y_T^\alpha = \partial_x g(X_T^\alpha)\,.$$

Note that under Assumption 4.9 and 2.3

$$\mathbb{E}[|\partial_x g(X_T^\alpha)|^2] \le \mathbb{E}[(K(1+|X_T^\alpha|)^2] < \infty,$$

Hence, due to Theorem 4.7, the adjoint BSDEs have unique solutions (Y^α, Z^α).

We will now see that it is possible to formulate a sufficient optimality criteria based on the properties of the Hamiltonian and based on the adjoint BSDEs. This is what is known as the *Pontryagin's Maximum Principle*. Consider two control processes, $\alpha, \beta \in \mathcal{U}$ and the two associated controlled diffusions, both starting from the same initial value, labelled X^α, X^β. Then

$$J(\beta) - J(\alpha) = \mathbb{E}\left[\int_0^T \left[f(t, X_t^\beta, \beta_t) - f(t, X_t^\alpha, \alpha_t)\right]\,dt + g(X_T^\beta) - g(X_T^\alpha)\right]\,.$$

We will need to assume that g is concave (equivalently assume $-g$ is convex). Then $g(x) - g(y) \ge \partial_x g(x)(x-y)$ and so (recalling what the terminal condition in our adjoint equation is)

$$\mathbb{E}\left[g(X_T^\beta) - g(X_T^\alpha)\right] \ge \mathbb{E}\left[(X_T^\beta - X_T^\alpha)\partial_x g(X_T^\beta)\right] = \mathbb{E}\left[(X_T^\beta - X_T^\alpha)Y_T^\beta\right]\,.$$

We use Itô's product rule and the fact that $X_0^\alpha = X_0^\beta$. Let us write $\Delta b_t := b_t(X_t^\beta, \beta_t) - b_t(X_t^\alpha, \alpha_t)$ and $\Delta\sigma_t := \sigma_t(X_t^\beta, \beta_t) - \sigma_t(X_t^\alpha, \alpha_t)$. Then we see that

$$\begin{aligned}\mathbb{E}\left[(X_T^\beta - X_T^\alpha)Y_T^\beta\right] \ge \mathbb{E}\bigg[&\int_0^T -(X_t^\beta - X_t^\alpha)\partial_x H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta)\,dt \\ &+ \int_0^T \Delta b_t\, Y_t^\beta\,dt + \int_0^T \mathrm{tr}\left[\Delta\sigma_t^\top\, Z_t^\beta\right]\,dt\bigg]\,.\end{aligned}$$

Note that we are missing some details here, because the second stochastic integral term that we dropped isn't necessarily a martingale. However with a stopping time argument and Fatou's Lemma the details can be filled in (and this is why we have an inequality). We also have that for all y, z,

$$\begin{aligned} f(t, X_t^\beta, \beta_t) &= H_t(X_t^\beta, \beta_t, y, z) - b_t(X_t^\beta, \beta_t)y - \mathrm{tr}[\sigma_t^\top(X_t^\beta, \beta_t)z]\,,\\ f(t, X_t^\alpha, \alpha_t) &= H_t(X_t^\alpha, \alpha_t, y, z) - b_t(X_t^\alpha, \alpha_t)y - \mathrm{tr}[\sigma_t^\top(X_t^\alpha, \alpha_t)z]\end{aligned}$$

and so

$$f(t, X_t^\beta, \beta_t) - f(t, X_t^\alpha, \alpha_t) = \Delta H_t - \Delta b_t Y_t^\beta - \mathrm{tr}(\Delta\sigma_t^\top\, Z_t^\beta)$$

where

$$\Delta H_t := H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta) - H_t(X_t^\alpha, \alpha_t, Y_t^\beta, Z_t^\beta)\,.$$

Thus

$$\mathbb{E}\left[\int_0^T \left[f(t, X_t^\beta, \beta_t) - f(t, X_t^\alpha, \alpha_t)\right]\,dt\right] = \mathbb{E}\left[\int_0^T \left[\Delta H_t - \Delta b_t Y_t^\beta - \mathrm{tr}(\Delta\sigma_t^\top\, Z_t^\beta)\right]\,dt\right]\,.$$

Altogether

$$J(\beta) - J(\alpha) \geq \mathbb{E}\left[\int_0^T \left[\Delta H_t - (X_t^\beta - X_t^\alpha)\partial_x H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta)\right] \, dt\right]$$

If we now assume that $(x, a) \mapsto H_t(x, a, Y_t^\beta, Z_t^\beta)$ is differentiable and concave for any t, y, z then

$$\Delta H_t \geq (X_t^\beta - X_t^\alpha)\partial_x H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta) + (\beta_t - \alpha_t)\partial_a H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta)$$

and so

$$J(\beta) - J(\alpha) \geq \mathbb{E}\left[\int_0^T (\beta_t - \alpha_t)\partial_a H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta) \, dt\right] .$$

Finally we assume that β_t is a control process which satisfies

$$H_t(X_t^\beta, \beta_t, Y_t^\beta, Z_t^\beta) = \max_{a \in A} H_t(X_t^\beta, a, Y_t^\beta, Z_t^\beta) < \infty \text{ a.s. for almost all } t. \tag{4.7}$$

Then $J(\beta) \geq J(\alpha)$ for arbitrary α. In other words, such control β is optimal. Hence we have proved the following theorem.

Theorem 4.10 (Pontryagin's Maximum Principle)**.** *Let Assumptions 2.3 and 4.9 holds, let $\subset \mathbb{R}^m$. Let g be concave. Let $\beta \in \mathcal{U}$ and let X^β be the associated controlled diffusion and (Y^β, Z^β) the solution of the adjoint BDSE. If $\beta \in \mathcal{U}$ is such that* (4.7) *holds and if*

$$(x, a) \mapsto H_t(x, a, Y_t^\beta, Z_t^\beta)$$

is differentiable and concave then $J(\beta) = \sup_\alpha J(\alpha)$ i.e. β is an optimal control.

We can see that the Pontryagin maximum principle gives us a sufficient condition for optimality.

Example 4.11 (Minimum variance for given expected return)**.** We consider the simplest possible model for optimal investment: we have a risk-free asset B with evolution given by $dB_t = rB_t \, dt$ and $B_0 = 1$ and a risky asset S with evolution given by $dS_t = \mu S_t \, dt + \sigma S_t \, dW_t$ with S_0 given. For simplicity we assume that σ, μ, r are given constants, $\sigma \neq 0$ and $\mu > r$. The value of a portfolio with no asset injections / consumption is given by $X_0 = x$ and

$$dX_t^\alpha = \frac{\alpha_t}{S_t} \, dS_t + \frac{X_t - \alpha_t}{B_t} \, dB_t \, ,$$

where α_t represents the amount invested in the risky asset. Then

$$dX_t^\alpha = (rX_t + \alpha_t(\mu - r)) \, dt + \sigma\alpha_t \, dW_t \, . \tag{4.8}$$

Given a desired return $m > 0$ we aim to find a trading strategy which would minimize the variance of the return (in other words a strategy that gets as close to the desired return as possible). We restrict ourselves to α such that $\mathbb{E}\int_0^T \alpha_t^2 \, dt < \infty$. Thus we seek

$$V(m) := \inf_\alpha \left\{\text{Var}(X_T^\alpha) : \mathbb{E}X_T^\alpha = m\right\} . \tag{4.9}$$

See Exercise 4.12 to convince yourself that the set over which we wish to take infimum is non-empty. Conveniently, if, for $\lambda \in \mathbb{R}$, we can calculate

$$v(\lambda) := \inf_\alpha \mathbb{E}\left[|X_T^\alpha - \lambda|^2\right]$$

then [Pha09, Proposition 6.6.5] tells us that

$$V(m) = \sup_{\lambda \in \mathbb{R}} \left[v(\lambda) - (m-\lambda)^2 \right].$$

Furthermore

$$v(\lambda) = -\sup_{\alpha} \mathbb{E}\left[-|X_T^\alpha - \lambda|^2 \right].$$

Thus our aim is to maximize

$$J_\lambda(\alpha) := \mathbb{E}\left[g(X_T^\alpha) \right] \text{ with } g(x) = -(x-\lambda)^2.$$

Since g is concave and differentiable we will try to apply Pontryagin's maximum principle. As there is no running gain (i.e. $f = 0$) and since X^α is given by (4.8) we have the Hamiltonian

$$H_t(x,a,y,z) = [rx + a(\mu - r)]y + \sigma a\, z.$$

This, being affine in (a,x), is certainly differentiable and concave. Moreover, if there is an optimal control β and if the solution of the adjoint BSDE is denoted (Y^β, Z^β) then

$$\max_a H_t(X_t^\beta, a, Y_t^\beta, Z_t^\beta) = \max_a \left[rX_t^\beta Y_t^\beta + a(\mu - r)Y_t^\beta + \sigma a Z_t^\beta \right].$$

The quantity being maximized is linear in a and thus it will be finite if and only if the solution to the adjoint equation satisfies

$$(\mu - r)Y_t^\beta + \sigma Z_t^\beta = 0 \text{ a.s. for a.a } t. \tag{4.10}$$

From now on we omit the superscript β everywhere. Recalling the adjoint equation:

$$dY_t = -rY_t\, dt + Z_t\, dW_t \text{ and } Y_T = \partial_x g(X_T) = -2(X_T - \lambda). \tag{4.11}$$

To proceed we will need to make a guess at what the solution to the adjoint BSDE will look like. Since the terminal condition is linear in X_T we will try the ansatz $Y_t = \varphi(t)X_t + \psi(t)$ for some C^1 functions φ and ψ. Notice that this is rather different to the situation in Example 4.3, since there we obtain a solution but only in terms of an unknown process arising from the martingale representation theorem. With this ansatz we have, substituting the expression for Y on the r.h.s. of (4.11), that

$$dY_t = -r\varphi(t)X_t\, dt - r\psi(t)\, dt + Z_t\, dW_t \tag{4.12}$$

and on the other hand we can use the ansatz for Y and product rule on the l.h.s. of (4.11) to see

$$\begin{aligned} dY_t &= \varphi(t)\, dX_t + X_t\varphi'(t)\, dt + \psi'(t)\, dt \\ &= \varphi(t)\left[rX_t + \beta_t(\mu - r) \right] dt + \varphi(t)\sigma\beta_t\, dW_t + X_t\varphi'(t)\, dt + \psi'(t)\, dt. \end{aligned} \tag{4.13}$$

The second equality above came from (4.8) with β as the control. Then (4.12) and (4.13) can simultaneously hold only if $Z_t = \varphi(t)\sigma\beta_t$ and if

$$\varphi(t)\left[rX_t + \beta_t(\mu - r) \right] + X_t\varphi'(t) + \psi'(t) = -r\varphi(t)X_t - r\psi(t).$$

This in turn will hold as long as

$$\beta_t = \frac{2r\varphi(t)X_t + r\psi(t) + \varphi'(t)X_t + \psi'(t)}{\varphi(t)(r - \mu)}. \tag{4.14}$$

On the other hand from the Pontryagin maximum principle we conculded (4.10) which, with $Y_t = \varphi(t)X_t + \psi(t)$ and $Z_t = \varphi(t)\sigma\beta_t$ says

$$(\mu - r)[\varphi(t)X_t + \psi(t)] + \sigma^2\varphi(t)\beta_t = 0,$$

i.e.

$$\beta_t = \frac{(r-\mu)[\varphi(t)X_t + \psi(t)]}{\sigma^2\varphi(t)}\,. \tag{4.15}$$

But (4.14) and (4.15) can both hold only if (collecting terms with X_t and without)

$$\begin{aligned} \varphi'(t) &= \left(\tfrac{(r-\mu)^2}{\sigma^2} - 2r\right)\varphi(t)\,, \ \ \varphi(T) = -2 \\ \psi'(t) &= \left(\tfrac{(r-\mu)^2}{\sigma^2} - r\right)\psi(t)\,, \ \ \psi(T) = 2\lambda\,. \end{aligned} \tag{4.16}$$

Note that the terminal conditions arose from Y_T (rather than from the equations for β). Also note that ψ clearly depends on λ but for now we omit this in our notation. Clearly

$$\varphi(t) = -2e^{-\left(\frac{(r-\mu)^2}{\sigma^2}-2r\right)(T-t)} \ \text{ and } \ \psi(t) = 2\lambda e^{-\left(\frac{(r-\mu)^2}{\sigma^2}-r\right)(T-t)}\,. \tag{4.17}$$

We note that from (4.15) we can write the control as Markov control

$$\beta(t,x) = -\frac{(\mu-r)[\varphi(t)x + \psi(t)]}{\sigma^2\varphi(t)}\,.$$

Thus X driven by this control is square integrable. Indeed β is a linear function in x and together with (4.8) and Proposition 1.28 we can conclude the square integrability. Thus we also have $\mathbb{E}\int_0^T \beta_t^2\,dt < \infty$ and so the control is admissable.

We still need to know

$$v(\lambda) = -J(\beta) = \mathbb{E}\left[|X_T - \lambda|^2\right]\,.$$

We cannot calculate this by solving for X as in Exercise 4.12 (try it). Instead we note that

$$\mathbb{E}|X_T - \lambda|^2 = \mathbb{E}\left[-\frac{1}{2}\varphi(T)X_T^2 - \psi(T)X_T + \lambda^2\right].$$

From Itô's formula for $\xi_t := -\frac{1}{2}\varphi(t)X_t^2 - \psi(t)X_t$ we get that

$$-d\xi_t = \left(\tfrac{1}{2}\varphi'(t)X_t^2 + \psi'(t)X_t\right)\,dt + [\varphi(t)X_t + \psi(t)]\,dX_t + \tfrac{1}{2}\varphi(t)\,dX(t)dX(t)\,.$$

And we have that

$$dX_t = (rX_t + \beta_t(\mu - r))\,dt + \sigma\beta_t\,dW_t\,.$$

Hence

$$\begin{aligned} -\mathbb{E}\xi_T = -\xi_0 + \mathbb{E}\int_0^T \Big(&\tfrac{1}{2}\varphi'(t)X_t^2 + \psi'(t)X_t \\ &+ r\varphi(t)X_t^2 + r\psi(t)X_t \\ &+ \beta_t(\mu - r)[\varphi(t)X_t + \psi(t)] \\ &+ \tfrac{1}{2}\varphi(t)\sigma^2\beta_t^2\Big)\,dt\,. \end{aligned}$$

From the optimality condition $(\mu - r)\beta_t[\varphi(t)X_t + \psi_t] + \sigma^2\varphi(t)\beta_t^2 = 0$ we get

$$\tfrac{1}{2}\sigma^2\varphi(t)\beta_t^2 = -\tfrac{1}{2}(\mu - r)\beta_t[\varphi(t)X_t + \psi_t]$$

and so

$$-\mathbb{E}\xi_T = -\xi_0 + \mathbb{E}\int_0^T \Big(\tfrac{1}{2}\varphi'(t)X_t^2 + \psi'(t)X_t + r\varphi(t)X_t^2 + r\psi(t)X_t + \tfrac{1}{2}\beta_t(\mu - r)[\varphi(t)X_t + \psi(t)]\Big)\,dt\,.$$

This is

$$-\mathbb{E}\xi_T = -\xi_0 + \tfrac{(r-\mu)^2}{\sigma^2}\mathbb{E}\int_0^T \left(\tfrac{1}{2}\varphi(t)X_t^2 + \psi(t)X_t - \tfrac{1}{2}\tfrac{\varphi(t)^2X_t^2 + 2\varphi(t)\psi(t)X_t + \psi(t)^2}{\varphi(t)}\right)dt\,.$$

So

$$\mathbb{E}\xi_T = \xi_0 + \tfrac{1}{2}\tfrac{(r-\mu)^2}{\sigma^2}\int_0^T \tfrac{\psi(t)^2}{\varphi(t)}\,dt\,.$$

Due to (4.17) we have

$$\mathbb{E}\xi_T = \xi_0 - \lambda^2\tfrac{(r-\mu)^2}{\sigma^2}\int_0^T e^{-\frac{(r-\mu)^2}{\sigma^2}(T-t)}\,dt\,.$$

Hence

$$\mathbb{E}\xi_T = \xi_0 - \lambda^2\left[1 - e^{-\frac{(r-\mu)^2}{\sigma^2}T}\right]\,.$$

But $\mathbb{E}|X_T - \lambda|^2 = \mathbb{E}\xi_T + \lambda^2$ and so

$$\mathbb{E}|X_T - \lambda|^2 = \xi_0 + \lambda^2 e^{-\frac{(r-\mu)^2}{\sigma^2}T}\,.$$

Moreover $\xi_0 = -\frac{1}{2}\varphi(0)x^2 - \psi(x)x$ and so

$$\xi_0 = x^2 e^{-\left(\frac{(r-\mu)^2}{\sigma^2} - 2r\right)T} - 2x\lambda e^{-\left(\frac{(r-\mu)^2}{\sigma^2} - r\right)T}\,.$$

Finally

$$\mathbb{E}|X_T - \lambda|^2 = e^{-\frac{(r-\mu)^2}{\sigma^2}T}\left[x^2e^{2rT} - 2x\lambda e^{rT} + \lambda^2\right] = e^{-\frac{(r-\mu)^2}{\sigma^2}T}\left(\lambda - xe^{rT}\right)^2\,.$$

which means that

$$v(\lambda) = -\kappa\left(\lambda - xe^{rT}\right)^2\,,$$

where $\kappa := e^{-\frac{(r-\mu)^2}{\sigma^2}T} > 0$. We thus get

$$V(m) = \sup_{\lambda\in\mathbb{R}}\left[-\kappa\left(\lambda^2 - 2\lambda xe^{rT} + x^2e^{2rT}\right) - \lambda^2 + 2\lambda m - m^2\right]\,.$$

This is achieved when

$$0 = -\kappa\lambda + \kappa xe^{rT} - \lambda + m$$

i.e. when $\lambda = \frac{\kappa xe^{rT} + m}{\kappa + 1}$.

4.3 Exercises

Exercise 4.12 (To complement Example 4.11)**.** Show that, under the assumptions of Example 4.11, the set $\{\text{Var}(X_T^\alpha) : \mathbb{E}X_T^\alpha = m\}$ is nonempty.

4.4 Solutions to the exercises

Solution (to Exercise 4.12). We start by solving (4.8) for some $\alpha_t = a$ constant. Note that (with $X = X^\alpha$)

$$d(e^{-rt}X_t) = e^{-rt}\left[dX_t - rX_t\,dt\right] = e^{-rt}\left[a(\mu - r)\,dt + \sigma a\,dW_t\right]\,.$$

Thus

$$e^{-rT}X_T = x + \int_0^T e^{-rt}a(\mu - r)\,dt + \int_0^T \sigma a e^{-rt}\,dW_t\,.$$

Since the stochastic integral is a true martingale

$$\mathbb{E}X_T = e^{rT}x + e^{rT}a(\mu - r)\int_0^T e^{-rt}\,dt = e^{rT}x + a(\mu - r)\frac{1}{r}\left(e^{rT} - 1\right)\,.$$

Thus with

$$a = r\frac{m - e^{rT}x}{(\mu - r)(e^{rT} - 1)}$$

we see that $\mathbb{E}X_T = m$ and so the set is non-empty.

A Appendix

A.1 Useful Results from Other Courses

The aim of this section is to collect, mostly without proofs, results that are needed or useful for this course but that cannot be covered in the lectures i.e. prerequisites. You are expected to be able to use the results given here.

A.1.1 Linear Algebra

The inverse of a square real matrix A exists if and only if $\det(A) \neq 0$.

The inverse of square real matricies A and B exists if and only if the inverse of AB exists and moreover $(AB)^{-1} = B^{-1}A^{-1}$.

The inverse of a square real matrix A exists if and only if the inverse of A^T exists and $(A^T)^{-1} = (A^{-1})^T$.

If x is a vector in $\mathbb{R}^d$ then diag(x) denotes the matrix in $\mathbb{R}^{d\times d}$ with the entries of x on its diagonal and zeros everywhere else. The inverse of diag(x) exists if and only if $x_i \neq 0$ for all $i = 1, \ldots, d$ and moreover

$$\text{diag}(x)^{-1} = \text{diag}(1/x_1, 1/x_2, \ldots, 1/x_d).$$

A.1.2 Real Analysis and Measure Theory

Let $(X, \mathcal{X}, \mu)$ be a measure space (i.e. X is a set, $\mathcal{X}$ a σ-algebra and μ a measure).

Lemma A.1 (Fatou's Lemma)**.** *Let $f_1, f_2, \ldots$ be a sequence of non-negative and measurable functions. Then the function defined point-wise as*

$$f(x) := \liminf_{k\to\infty} f_k(x)$$

is $\mathcal{X}$-measurable and

$$\int_X f\, d\mu \leq \liminf_{k\to\infty} \int_X f_k\, d\mu.$$

Consider sets X and Y with σ-algebras $\mathcal{X}$ and $\mathcal{Y}$. By $\mathcal{X} \times \mathcal{Y}$ we denote the collection of sets $C = A \times B$ where $A \in \mathcal{X}$ and $B \in \mathcal{Y}$. By $\mathcal{X} \otimes \mathcal{Y} = \sigma(\mathcal{X} \times \mathcal{Y})$, which is the σ-algebra generated by $\mathcal{X} \times \mathcal{Y}$.

Theorem A.2. *Let $f : X \times Y \to \mathbb{R}$ be a measurable function, i.e. measurable with respect to the σ-algebras $\mathcal{X} \otimes \mathcal{Y}$ and $\mathcal{B}(\mathbb{R})$. Then for each $x \in X$ the function $y \mapsto f(x, y)$ is measurable with respecto to $\mathcal{Y}$ and $\mathcal{B}(\mathbb{R})$. Similarly for each $y \in Y$ the function $x \mapsto f(x, y)$ is measurable with respecto to $\mathcal{X}$ and $\mathcal{B}(\mathbb{R})$.*

The proof is short and so it's easiest to just include it here.

Proof. We first consider functions of the form $f = \mathbb{1}_C$ with $C \in \mathcal{X} \otimes \mathcal{Y}$. Let

$$\mathcal{H} = \{C \in \mathcal{X} \otimes \mathcal{Y} : y \mapsto \mathbb{1}_C(x, y) \text{ is } \mathcal{F} - \text{measurable for each fixed } x \in E\}.$$

It is easy to check that $\mathcal{H}$ is a σ-algebra. Moreover if $C = A \times B$ with $A \in \mathcal{X}$ and $B \in \mathcal{Y}$ then

$$y \mapsto \mathbb{1}_C(x,y) = \mathbb{1}_A(x)\mathbb{1}_B(y).$$

As x is fixed $\mathbb{1}_A(x)$ is just a constant and since $B \in \mathcal{Y}$ the function $y \mapsto \mathbb{1}_A(x)\mathbb{1}_B(y)$ must be measurable. Hence $\mathcal{X} \times \mathcal{Y} \subseteq \mathcal{H}$ and thus $\mathcal{X} \otimes \mathcal{Y} \subseteq \mathcal{H}$. But $\mathcal{H} \subseteq \mathcal{X} \otimes \mathcal{Y}$ and so $\mathcal{H} = \mathcal{X} \otimes \mathcal{Y}$. Hence if f is a simple function then the conclusion of the theorem holds.

Now consider $f \geq 0$ and let f_n be a sequence of simple functions increasing to f. Then for a fixed x the function $y \mapsto g_n(y) = f_n(x,y)$ is measurable. Moreover since $g(y) = \lim_{n\to\infty} g_n(y) = f(x,y)$ and since the limit of measurable functions is measurable we get the result for $f \geq 0$. For general $f = f^+ - f^-$ the result follows using the result for $f^+ \geq 0, f^- \geq 0$ and noting that the difference of measurable functions is measurable. $\square$

Consider measure spaces $(X, \mathcal{X}, \mu_x)$, $(Y, \mathcal{Y}, \mu_y)$. That is, X and Y are sets, $\mathcal{X}$ and $\mathcal{Y}$ are σ-algebras and μ_x and μ_y are measures on $\mathcal{X}$ and $\mathcal{Y}$ respectively. For all details on Fubini's Theorem we refer to Kolmogorov and Fomin [?kolmogorov:fomin:real].

Theorem A.3 (Fubini). *Let μ be the Lebesgue extension of $\mu_x \otimes \mu_y$. Let $A \in \mathcal{X} \otimes \mathcal{Y}$. and let $f : A \to \mathbb{R}$ be a measurable function (considering $\mathcal{B}(\mathbb{R})$, the Borel σ-algebra on $\mathbb{R}$). If f is integrable i.e. if*

$$\int_A |f(x,y)| d\mu < \infty$$

then

$$\int_A f(x,y) d\mu = \int_X \left[\int_{A_x} f(x,y) d\mu_y \right] d\mu_x = \int_Y \left[\int_{A_y} f(x,y) d\mu_x \right] d\mu_y,$$

where $A_x := \{y \in Y : (x,y) \in A\}$ and $A_y := \{x \in X : (x,y) \in A\}$.

Remark A.4. The conclusion of Fubini's theorem implies that for μ_x-almost all x the integral $\int_{A_x} f(x,y) d\mu_y$ exists which in turn implies that the function $f(x,\cdot) : A_x \to \mathbb{R}$ must be measurable. This statement also holds if we exchange x for y.

A.1.3 Conditional Expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be given.

Theorem A.5. *Let X be an integrable random variable. If $\mathcal{G} \subseteq \mathcal{F}$ is a σ-algebra then there exists a unique $\mathcal{G}$ measurable random variable Z such that*

$$\forall G \in \mathcal{G} \quad \int_G X d\mathbb{P} = \int_G Z d\mathbb{P}.$$

The proof can be found in xxxx xxxx.

Definition A.6. Let X be an integrable random variable. If $\mathcal{G} \subseteq \mathcal{F}$ is a σ-algebra then $\mathcal{G}$-random variable from Theorem A.5 is called the *conditional expectation* of X given $\mathcal{G}$ and write $\mathbb{E}(X|\mathcal{G}) := Z$.

Conditional expectations are rather abstract notion so two examples might help.

Example A.7. Consider $\mathcal{G} := \{\emptyset, \Omega\}$. So $\mathcal{G}$ is just the trivial σ-algebra. For a random variable X we then have, by definition, that Z is the conditional expectation (denoted $\mathbb{E}[X|\mathcal{G}]$), if and only if

$$\int_\Omega Z d\mathbb{P} = \int_\Omega X d\mathbb{P}.$$

The right hand side of the above expression is in fact just $\mathbb{E}X$ and so the equality would be satisfied if we set $Z = \mathbb{E}X$ (just a constant). Indeed then (going right to left)

$$\mathbb{E}X = \int_\Omega X d\mathbb{P} = \int_\Omega Z d\mathbb{P} = \int_\Omega \mathbb{E}X d\mathbb{P} = \mathbb{E}X \int_\Omega d\mathbb{P} = \mathbb{E}X.$$

Example A.8. Let $X \sim N(0,1)$. Let $\mathcal{G} = \{\emptyset, \{X \leq 0\}, \{X > 0\}, \Omega\}$. One can (and should) check that this is a σ-algebra. By definition the conditional expectation is a unique random variable that satisfies

$$\begin{aligned} \int_\Omega \mathbb{1}_{\{X>0\}} Z d\mathbb{P} &= \int_\Omega \mathbb{1}_{\{X>0\}} X d\mathbb{P}, \\ \int_\Omega \mathbb{1}_{\{X\leq 0\}} Z d\mathbb{P} &= \int_\Omega \mathbb{1}_{\{X\leq 0\}} X d\mathbb{P}, \\ \int_\Omega Z d\mathbb{P} &= \int_\Omega X d\mathbb{P}. \end{aligned} \tag{A.1}$$

It is a matter of integrating with respect to normal density to find out that

$$\int_\Omega \mathbb{1}_{\{X>0\}} X d\mathbb{P} = \int_0^\infty x\phi(x)dx = \frac{1}{2}\sqrt{\frac{2}{\pi}}, \quad \int_\Omega \mathbb{1}_{\{X\leq 0\}} X d\mathbb{P} = -\frac{1}{2}\sqrt{\frac{2}{\pi}}. \tag{A.2}$$

Since Z must be $\mathcal{G}$ measurable it can only take two values:

$$Z = \begin{cases} z_1 & \text{on} \quad \{X > 0\}, \\ z_2 & \text{on} \quad \{X \leq 0\}, \end{cases}$$

for some real constants z_1 and z_2 to be yet determined. But (A.1) and (A.2) taken together imply that

$$\frac{1}{2}\sqrt{\frac{2}{\pi}} = \int_\Omega \mathbb{1}_{\{X>0\}} Z d\mathbb{P} = \int_\Omega \mathbb{1}_{\{X>0\}} z_1 d\mathbb{P} = z_1 \mathbb{P}(X > 0) = \frac{1}{2} z_1.$$

Hence $z_1 = \sqrt{2/\pi}$. Similarly we calculate that $z_2 = -\sqrt{2/\pi}$. Finally we check that the third equation in (A.1) holds. Thus

$$\mathbb{E}[X|\mathcal{G}] = Z = \begin{cases} \sqrt{\frac{2}{\pi}} & \text{on} \quad \{X > 0\}, \\ -\sqrt{\frac{2}{\pi}} & \text{on} \quad \{X \leq 0\}. \end{cases}$$

Here are some further important properties of conditional expectations which we present without proof.

Theorem A.9 (Properties of conditional expectations)**.** *Let X and Y be random variables. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$.*

1. *If $\mathcal{G} = \{\emptyset, \Omega\}$ then $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}X$.*
2. *If $X = x$ a. s. for some constant $x \in \mathbb{R}$ then $\mathbb{E}(X|\mathcal{G}) = x$ a.s..*

3. *For any* $\alpha, \beta \in \mathbb{R}$

$$\mathbb{E}(\alpha X + \beta Y|\mathcal{G}) = \alpha\mathbb{E}(X|\mathcal{G}) + \beta\mathbb{E}(Y|\mathcal{G}).$$

   This is called linearity.

4. *If* $X \leq Y$ *almost surely then* $\mathbb{E}(X|\mathcal{G}) \leq \mathbb{E}(Y|\mathcal{G})$*a.s.* .
5. $|\mathbb{E}(X|\mathcal{G})| \leq \mathbb{E}(|X|\,|\mathcal{G})$.
6. *If* $X_n \to X$ *a. s. and* $|X_n| \leq Z$ *for some integrable* Z *then* $\mathbb{E}(X_n|\mathcal{G}) \to \mathbb{E}(X|\mathcal{G})$ *a. s. . This is the "dominated convergence theorem for conditional expectation".*
7. *If* Y *is* $\mathcal{G}$ *measurable then* $\mathbb{E}(XY|\mathcal{G}) = Y\mathbb{E}(X|\mathcal{G})$.
8. *Let* $\mathcal{H}$ *be a sub-*σ*-algebra of* $\mathcal{G}$*. Then*

$$\mathbb{E}(X|\mathcal{H}) = \mathbb{E}(\mathbb{E}(X|\mathcal{G})|\mathcal{H}).$$

   This is called the tower property. *A special case is* $\mathbb{E}X = \mathbb{E}(\mathbb{E}(X|\mathcal{G}))$.

9. *If* $\sigma(X)$ *is independent of* $\mathcal{G}$ *then* $\mathbb{E}(X|\mathcal{G}) = \mathbb{E}X$.

Definition A.10. Let X and Y be two random variables. The *conditional expectation of* X *given* Y is defined as $\mathbb{E}(X|Y) := \mathbb{E}(X|\sigma(Y))$, that is, it is the conditional expectation of X given the σ-algebra generated by Y.

Definition A.11. Let X a random variables and $A \in \mathcal{F}$ an event. The *conditional expectation of* X *given* A is defined as $\mathbb{E}(X|A) := \mathbb{E}(X|\sigma(A))$. This means it is the conditional expectation of X given the sigma algebra generated by A i.e. $\mathbb{E}(X|A) := \mathbb{E}(X|\{\emptyset, A, A^c, \Omega\})$.

We can immediately see that $\mathbb{E}(X|A) = \mathbb{E}(X|\mathbb{1}_A)$.

Recall that if X and Y are jointly continuous random variables with joint density $(x, y) \mapsto f(x, y)$ then for any measurable function $\rho : \mathbb{R}^2 \to \mathbb{R}$ such that $\mathbb{E}|\rho(X, Y)| < \infty$ we have

$$\mathbb{E}\rho(X, Y) = \int_{\mathbb{R}}\int_{\mathbb{R}} \rho(x, y)f(x, y)dydx.$$

Moreover the marginal density of X is

$$g(x) = \int_{\mathbb{R}} f(x, y)dy$$

while the marginal density of Y is

$$h(y) = \int_{\mathbb{R}} f(x, y)dx.$$

Theorem A.12. *Let* X *and* Y *be jointly continuous random variables with joint density* $(x, y) \mapsto f(x, y)$*. Then for any measurable function* $\varphi : \mathbb{R} \to \mathbb{R}$ *such that* $\mathbb{E}|\varphi(Y)| < \infty$ *the conditional expectation of* $\varphi(Y)$ *given* X *is*

$$\mathbb{E}(\varphi(Y)|X) = \psi(X)$$

where $\psi : \mathbb{R} \to \mathbb{R}$ *is given by*

$$\psi(x) = \mathbb{1}_{\{g(x)>0\}} \frac{\int_{\mathbb{R}} \varphi(y)f(x, y)dy}{g(x)}.$$

Proof. Every A in $\sigma(X)$ must be of the form $A = \{\omega \in \Omega : X(\omega) \in B\}$ for some B in $\mathcal{B}(\mathbb{R})$. We need to show that for any such A

$$\int_A \psi(X) d\mathbb{P} = \int_A \varphi(Y) d\mathbb{P}.$$

But since $\mathbb{E}|\varphi(Y)| < \infty$ we can use Fubini's theorem to show that

$$\begin{aligned}
&\int_A \psi(X) d\mathbb{P} = \mathbb{E}\mathbb{1}_A \psi(X) = \mathbb{E}\mathbb{1}_{\{X \in B\}} \psi(X) = \int_B \psi(x) g(x) dx \\
&= \int_B \int_{\mathbb{R}} \varphi(y) f(x,y) dy dx = \int_{\mathbb{R}} \int_{\mathbb{R}} \mathbb{1}_B(x) \varphi(y) f(x,y) dx dy \\
&= \mathbb{E}\mathbb{1}_{\{X \in B\}} \varphi(Y) = \int_A \varphi(Y) d\mathbb{P}.
\end{aligned}$$

□

Let on $(\Omega, \mathcal{F})$ be a measurable space. Recall that we say that a measure $\mathbb{Q}$ is absolutely continuous with respect to a measure $\mathbb{P}$ if $\mathbb{P}(E) = 0$ implies that $\mathbb{Q}(E) = 0$. We write $\mathbb{Q} << \mathbb{P}$.

Proposition A.13. *Take two probability measures $\mathbb{P}$ and $\mathbb{Q}$ such that $\mathbb{Q} << \mathbb{P}$ with*

$$d\mathbb{Q} = \Lambda d\mathbb{P}.$$

Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Then $\mathbb{Q}$ almost surely $\mathbb{E}[\Lambda|\mathcal{G}] > 0$. Moreover for any $\mathcal{F}$-random variable X we have

$$\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] = \frac{\mathbb{E}[X\Lambda|\mathcal{G}]}{\mathbb{E}[\Lambda|\mathcal{G}]}. \tag{A.3}$$

Proof. Let $S := \{\omega : \mathbb{E}[\Lambda|\mathcal{G}](\omega) = 0\}$. Then $S \in \mathcal{G}$ and so by definition of conditional expectation

$$\mathbb{Q}(S) = \int_S d\mathbb{Q} = \int_S \Lambda d\mathbb{P} = \int_S \mathbb{E}[\Lambda|\mathcal{G}] d\mathbb{P} = \int_S 0 \, d\mathbb{P} = 0.$$

Thus $\mathbb{Q}$-a.s. we have $\mathbb{E}[\Lambda|\mathcal{G}](\omega) > 0$.

To prove the second claim assume first that $X \geq 0$. We note that by definition of conditional expectation, for all $G \in \mathcal{G}$:

$$\int_G \mathbb{E}[X\Lambda|\mathcal{G}] d\mathbb{P} = \int_G X\Lambda d\mathbb{P} = \int_G X d\mathbb{Q} = \int_G \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] d\mathbb{Q} = \int_G \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] \Lambda d\mathbb{P}.$$

Now we use the definition of conditional expectation to take *another* conditional expectation with respect to $\mathcal{G}$. Since $G \in \mathcal{G}$:

$$\int_G \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] \Lambda d\mathbb{P} = \int_G \mathbb{E}\left[\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\Lambda|\mathcal{G}\right] d\mathbb{P}.$$

But $\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]$ is $\mathcal{G}$-measurable and so

$$\int_G \mathbb{E}\left[\mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\Lambda|\mathcal{G}\right] d\mathbb{P} = \int_G \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}] \mathbb{E}\left[\Lambda|\mathcal{G}\right] d\mathbb{P}.$$

Thus, since in particular $\Omega \in \mathcal{G}$, we get

$$\int_\Omega \mathbb{E}[X\Lambda|\mathcal{G}]d\mathbb{P} = \int_\Omega \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\mathbb{E}\left[\Lambda|\mathcal{G}\right]d\mathbb{P}.$$

Since $X \geq 0$ (and $\Lambda \geq 0$) this means that $\mathbb{P}$-a.s. and hence $\mathbb{Q}$-a.s. we have (A.3).

$$\mathbb{E}[X\Lambda|\mathcal{G}] = \mathbb{E}^{\mathbb{Q}}[X|\mathcal{G}]\mathbb{E}\left[\Lambda|\mathcal{G}\right].$$

For a general X write $X = X^+ - X^-$, where $X^+ = \mathbb{1}_{\{X\geq 0\}}X \geq 0$ and $X^- = -\mathbb{1}_{\{X<0\}}X \geq 0$. Then

$$\mathbb{E}^{\mathbb{Q}}[X^+ - X^-|\mathcal{G}] = \frac{\mathbb{E}[X^+\Lambda|\mathcal{G}]}{\mathbb{E}[\Lambda|\mathcal{G}]} - \frac{\mathbb{E}[X^-\Lambda|\mathcal{G}]}{\mathbb{E}[\Lambda|\mathcal{G}]} = \frac{\mathbb{E}[X^+ - X^-\Lambda|\mathcal{G}]}{\mathbb{E}[\Lambda|\mathcal{G}]}.$$

□

A.1.4 Multivariate normal distribution

There are a number of ways how to define a multivariate normal distribution. See e.g. [?gut:intermediate, Chapter 5] for a more definite treatment. We will define a multivariate normal distribution as follows. Let $\mu \in \mathbb{R}^d$ be given and let Σ be a given symmetric, invertible, positive definite $d \times d$ matrix (it is also possible to consider positive semi-definite matrix Σ but for simplicity we ignore that situation here).

A matrix is positive definite if, for any $x \in \mathbb{R}^d$ such that $x \neq 0$, the inequality $x^T\Sigma x > 0$ holds. From linear algebra we know that this is equivalent to:

1. The eigenvalues of the matrix Σ are all positive.
2. There is a unique (up to multiplication by -1) matrix B such that $BB^T = \Sigma$.

Let B be a $d \times k$ matrix such that $BB^T = \Sigma$.

Let $(X_i)_{i=1}^d$ be independent random variables with $N(0,1)$ distribution. Let $X = (X_1, \ldots, X_d)^T$ and $Z := \mu + BX$. We then say $Z \sim N(\mu, \Sigma)$ and call Σ the covariance matrix of Z.

Exercise A.14. Show that $\text{Cov}(Z_i, Z_j) = \mathbb{E}((Z_i - \mathbb{E}Z_i)(Z_j - \mathbb{E}Z_j)) = \Sigma_{ij}$. This justifies the name "covariance matrix" for Σ.

It is possible to show that the density function of $N(\mu, \Sigma)$ is

$$f(x) = \frac{1}{(2\pi)^{d/2}\sqrt{\det(\Sigma)}}\exp\left(-\frac{1}{2}((x-\mu)^T\Sigma^{-1}(x-\mu))\right). \tag{A.4}$$

Note that if Σ is symmetric and invertible then Σ^{-1} is also symmetric.

Exercise A.15. You will show that $Z = BX$ defined above has the density f given by (A.4) if $\mu = 0$.

i) Show that the characteristic function of $Y \sim N(0,1)$ is $t \mapsto \exp(-t^2/2)$. In other words, show that $\mathbb{E}(e^{itY}) = \exp(-t^2/2)$. *Hint.* complete the squares.

ii) Show that the characteristic function of a random variable Y with density f given by (A.4) is
$$\mathbb{E}\left(e^{i(\Sigma^{-1}\xi)^T Y}\right) = \exp\left(-\frac{1}{2}\xi^T\Sigma^{-1}\xi\right).$$
By taking $y = \Sigma^{-1}\xi$ conclude that
$$\mathbb{E}\left(e^{iy^T Y}\right) = \exp\left(-\frac{1}{2}y^T\Sigma^{-1}y\right).$$
Hint. use a similar trick to completing squares. You can use the fact that since Σ^{-1} is symmetric $\xi^T\Sigma^{-1}x = (\Sigma^{-1}\xi)^T x$.

iii) Recall that two distributions are identiacal if and only if their characteristic functions are identical. Compute $\mathbb{E}\left(e^{iy^T Z}\right)$ for $Z = BX$ and $X = (X_1, \ldots, X_d)^T$ with $(X_i)_{i=1}^d$ independent random variables such that $X_i \sim N(0,1)$. Hence conclude that Z has density given by (A.4) with $\mu = 0$.

You can now also try to show that all this works with $\mu \neq 0$.

A.1.5 Stochastic Analysis Details

The aim of this section is to collect technical details in stochastic analysis needed to make the main part of the notes correct but perhaps too technical to be of interest to many readers.

Definition A.16. We say that a process X is called *progressively measurable* if the function $(\omega, t) \mapsto X(\omega, t)$ is measurable with respect to $\mathcal{F}_t \otimes \mathcal{B}([0,t])$ for all $t \in [0,T]$.

We will use Prog_T to denote the σ-algebra generated by all the progressively measurable processes on $\Omega \times [0,T]$.

If X is progressively measurable then the processes $\left(\int_0^t X(s)ds\right)_{t\in[0,T]}$ and $(X(t \wedge \tau))_{t\in[0,T]}$ are adapted (provided the paths of X are Lebesgue integrable and provided τ is a stopping time). The important thing for us is that any left (or right) continuous adapted process is progressively measurable.

A.1.6 More Exercises

Exercise A.17. Say $f : \mathbb{R} \to \mathbb{R}$ is smooth and $W = (W(t))_{t\in[0,T]}$ is a Wiener process. Calculate
$$\mathbb{E}\left[f'(W(T))W(T)\right].$$

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